The model companion for set theory

Matteo Viale (joint with Giorgio Venturi)

Dipartimento di Matematica
Università di Torino

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Model completeness

Definition (Robinson, [2] Def. 3.2.6)

An $\mathcal{L}$-theory T is *model complete* if for any $\mathcal{M}, \mathcal{N}$ models of T with $\mathcal{M}$ a substructure of $\mathcal{N}$ we have that $\mathcal{M} \prec \mathcal{N}$.

Model completeness is an indication of "tameness" of the theory:

1. If T admits quantifier elimination, it is model complete.
2. If T is model complete and $\mathcal{M}$ models T, $T \cup \mathrm{Diag}(\mathcal{M})$ is complete.
3. If T is model complete, a set of axioms for T is given by its Π_2-fragment $T_{\forall\exists}$ [2, Lemma 3.2.12, Thm. 3.2.14].
4. If T has Skolem functions for its existential formulae, it is model complete [2, Thm. 5.1.8, Proof of 5.1.9].

In particular every theory T has a conservative extension to a theory T' which is model complete. Model completeness is not a fully satisfactory notion of "tameness", contrary to stability or simplicity for example.

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Theorem (Robinson's test, [2] Lemma 3.2.7)

The following are equivalent for an $\mathcal{L}$-theory T:

1. *T is model complete.*
2. *Every T-model $\mathcal{M}$ is existentially closed (i.e. for every superstructure $\mathcal{N}$ of $\mathcal{M}$, $\mathcal{M}$ is Σ_1-elementary in $\mathcal{N}$).*
3. *Every sentence of $\mathcal{L}$ is T-equivalent to a universal sentence.*

The standard example of a model complete theory is the theory of algebraically closed fields.

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Model companions

Definition (Robinson, [2] Def. 3.2.8)

Let T be an $\mathcal{L}$-theory. S is the *model companion* of T if

- S is model complete,
- for every T-model $\mathcal{M}$, there is an S-model $\mathcal{N}$ such that $\mathcal{M}$ embeds into $\mathcal{N}$; and conversely.

Standard example of a theory T with model companion S are:

- T the theory of commutative rings with no zero-divisors,
- S the theory of algebraically closed fields.

Remark

Different theories can have the same model companions:
S is also the model companion of the theory of fields.

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Theorem ([2] Lemmas 3.2.12, 3.2.13, Thm. 3.2.14)

Let $\mathcal{L}$ be a first order language, and S be an $\mathcal{L}$-theory consisting of sentences. The following holds:

1. *S admits at most one model companion T.*
2. *If S admits a model companion T, then*
   - $T_\forall = S_\forall$,
   - *T is the theory of all existentially closed structures of $T_\forall$,*
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$T_\forall$ denotes the fragment given by universal sentences of T, similarly we define $T_\exists$, $T_{\forall\exists}$, etc.

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A correct first order language for set theory

It is now time to confront myself with the following statement contradicting my abstract......

Theorem (Hirschfeld, 1976)

ZF *(and so also* ZFC) *has a model companion, which is the theory*

$$S = \{\forall x_1 \dots \forall x_n (x_1 \notin x_2 \vee x_2 \notin x_3 \vee \dots \vee x_{n-1} \notin x_n \vee x_n \notin x_1) : n \in \mathbb{N}\}.$$

S is certainly not informative on set theory (it just says that the graph of $\in$ has no loops).

In the language $\mathcal{L} = \{\in, =\}$, bounded formulae are not recognized by ZF as simple. For example:

$\forall x \in z \exists y \in z \phi(x, y)$ with ϕ quantifier-free, is a Π_2-formulae; for suitably chosen ϕ there is no way we can prove it equivalent to a Π_1-formula.

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We consider bounded formulae as simpler than Δ_1-properties, hence we should adjust our syntax to accomodate this.

Syntactic solution

Given the language

$$\mathcal{L}^* = \left\{ R_\phi : \phi \text{ bounded formula in } \{\in, =\} \right\},$$

ZFC* is the $\mathcal{L}^*$-theory obtained adding to ZFC the axiom

$$\forall \overline{x} \phi(\overline{x}) \leftrightarrow R_\phi(\vec{x})$$

for any bounded formula $\phi(\overline{x})$.

It is much more informative to look at the model companions for (completions of) ZFC*.

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The generic multiverse of set theory

Given that forcing is the most effective method to produce independence result, set theory scholars focused on the class of models that can be obtained by forcing, the generic multiverse:

$$\Omega(V) = \{H_\kappa^{V[G]} : V[G] \text{ is a forcing extension of } V, \\ \kappa \text{ is a regular cardinal in } V[G] \text{ with } G \subseteq H_\kappa\}.$$

Remark

The conditions on G and κ guarantee that we chose κ large enough so to be able to read off the poset $P \in V$ for which G is a V-generic filter: it is some poset $P \subseteq H_\kappa$ preserving the regularity of κ in $V[G]$.

$\Omega(V)$ is sloppily defined: There are no V-generic filters G for non-trivial (i.e. separative) forcing notion $P \in V$! In particular on the face of the above definition:

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There are two ways to solve this logical difficulty. The standard one is the following:

Definition

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$\Omega(M) \in V$ is a set, since all its elements are subsets of $V_{M \cap \mathrm{Ord}}$.

Again the second and third conditions on G and κ grant that one can read off for which $P \in M$ G is an M-generic filter: it is some $P \subseteq H_\kappa^M$ preserving the regularity of κ in $M[G]$.

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The generic multiverse of set theory

It is indeed the case that:

$H_\kappa^{M[G]}$ is a substructure of $H_\delta^{M[H]}$ if and only if

$$(\kappa \leq \delta \quad \text{and} \quad M[G] \subseteq M[H]).$$

This occurs if and only if $M[H]$ is obtained as a generic extension of $M[G]$ by a forcing $P \in M[G]$ if and only if $G \in M[H]$.

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I prefer this second option:

Notation

Given a complete boolean algebra B *and a cardinal* κ *such that* $[\![\kappa$ *is a regular cardinal*$]\!]_{\mathsf{B}} = 1_{\mathsf{B}}$ *and* B *has a dense subset of size at most* κ*, let*

$$H_\kappa^{\mathsf{B}} = \left\{\tau \in V^{\mathsf{B}} : [\![\tau \text{ has transitive closure of size less than } \check{\kappa}]\!]_{\mathsf{B}} = 1_{\mathsf{B}}\right\}$$

Definition

$$\Omega(V) = \{H_\kappa^{\mathsf{B}}/_G : \mathsf{B} \in V \text{ is a complete boolean algebra, } G \in \mathrm{St}(\mathsf{B})\}.$$

Provided one is familiar with the basics of boolean valued semantics (I can expand on this here), $\Omega(V)$ is a perfectly well-defined family of (non-standard) set-sized structures in the language of set theory.

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What kind of morphisms in the language $\mathcal{L} = \{\in, =\}$ between structures in $\Omega(V)$ we should consider?

The natural answers are: "the morphisms induced by forcing extensions", or "the standard morphism of $\mathcal{L}$-structures", where $\mathcal{L} = \{\in, \subseteq, =\}$.
The two points of view express the same family of morphisms:

Definition

$(H_\kappa^{\mathsf{B}}/_G, \in_{\mathsf{B}} /_G, =)$ is a substructure of $(H_\delta^{\mathsf{C}}/_H, \in_{\mathsf{C}} /_H, =)$ if and only if:

- $\kappa \leq \delta$,
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in which case the map $[\tau]_G \mapsto [\tau]_H$ defines a morphism of $V^{\mathsf{B}}/_G$ into $V^{\mathsf{C}}/_H$ whose restriction embeds $H_\kappa^{\mathsf{B}}/_G$ into $H_\delta^{\mathsf{C}}/_H$, and preserves Δ_1-properties.

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Model companions for ZFC* in the generic multiverse

Let $\dot\omega_1$ denote a canonical B-name such that

$$[\![\dot\omega_1 \text{ is the first uncountable cardinal}]\!]_{\mathsf{B}} = 1_{\mathsf{B}}.$$

Our analysis of the generic multiverse will outline that:

- $\mathrm{Th}(V)_\forall = \mathrm{Th}(M)_\forall$ for all $M \in \Omega(V)$, even when considering $\mathcal{L}^*$-universal sentences with parameters in H_{ω_1} (i.e we have an unlimited use of bounded quantification).
- Any $\mathcal{L}^*$-structure of the form $H^{\mathsf{B}}_{\dot\omega_1}/_G$ is $\mathcal{L}^*$-existentially closed with respect to its superstructures in $\Omega(V)$. A
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- Woodin's generic absoluteness results brings that

  The $\mathcal{L}^*$-structures of the form $H^{\mathsf{B}}_{\omega_1}/_G$ are all models of the theory $\mathrm{Th}(H^V_{\omega_1})$ with parameters, and $\mathrm{Th}(H^V_{\omega_1})$ is model complete with respect to $\Omega(V)$.

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Therefore if in V there are sufficiently many large cardinals:

$\mathrm{Th}(H_{\omega_1}^V)$ is the model companion of the theory of $\mathrm{Th}(V)_\forall$ with respect to the structures in $\Omega(V)$.

We have that:

- $\mathrm{Th}(V)_\forall$ with parameters in H_{ω_1} is true in all structures of the generic multiverse $\Omega(V)$.
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Are Th(V) and Th($H_{\omega_1}^V$) really model companions?

Problems

- There are many more models of Th$(V)_\forall$ then there are in $\Omega(V)$. Is it always the case that we can embed some model of Th$(V)_\forall$ into a model of Th$(H_{\omega_1}^V)_\forall$ and conversely? (We can indeed show that this is the case, BUT.....)
- Th($H_{\omega_1}^V$) is not model complete.

Solution

A second syntactic stretch of the semantic of set theory:

- Enlarge Th($H_{\omega_1}^V$) to a natural expansion T' of the theory of $H_{\omega_1}^V$ with more predicates, so that T' is model complete and has the same Π_1-theory of V.
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Second order number theory compared to $\text{Th}(H^V_{\omega_1})$

Second order number theory can be defined as the theory of the structure

$$(\mathcal{P}(\mathbb{N}) \cup \mathbb{N}, \in, \subseteq, =, \mathbb{N}).$$

Any definable (with parameters) subset of this structure is also definable in the structure

$$(H_{\omega_1}, \overline{R}_\phi : \phi \text{ a bounded formula}) :$$

$\mathbb{N} \in H_{\omega_1}$ and $\mathcal{P}(\mathbb{N})$ is a class definable by the bounded formula $(x \subseteq \mathbb{N})$ in H_{ω_1}.

$$(\overline{R}_\phi = \{\overline{a} : H_{\omega_1} \models \phi(a)\})$$

The converse also holds.

Notation

Π^1_n-sets (respectively Σ^1_n, Δ^1_n) are the subsets of $\mathcal{P}(\mathbb{N}) \equiv 2^{\mathbb{N}}$ defined by a Π_n-formula (respectively by a Σ_n-formula, at the same time by a Σ_n-formula and a Π_n-formula in $\mathcal{L}^*$).

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Notation

Π^1_n-sets (respectively Σ^1_n, Δ^1_n) are the subsets of $\mathcal{P}(\mathbb{N}) \equiv 2^{\mathbb{N}}$ defined by a Π_n-formula (respectively by a Σ_n-formula, at the same time by a Σ_n-formula and a Π_n-formula in $\mathcal{L}^*$).

Second order number theory compared to $\mathrm{Th}(H_{\omega_1}^V)$

Definition

Given $a \in H_{\omega_1}$, $r \in 2^{\mathbb{N}}$ codes a, if (modulo a recursive bijection of $\mathbb{N}$ with $\mathbb{N}^2$) we have that r codes a well founded extensional relation on $\mathbb{N}$ whose transitive collapse is the transitive closure of $\{a\}$.

We let:

- $\mathrm{Cod} : 2^{\mathbb{N}} \to H_{\omega_1}$ be the map assigning a to r if and only if r codes a and assigning the emptyset to r otherwise.
- WFE be the set of $r \in 2^{\mathbb{N}}$ which (modulo a recursive bijection of $\mathbb{N}$ with $\mathbb{N}^2$) are a well founded extensional relation.

Remark

The map Cod is defined by a provably Δ_1-property over H_{ω_1} and is surjective. Moreover WFE is a Π^1_1-subset of $2^{\mathbb{N}}$.

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Second order number theory is not model complete.

Equivalence of second order number theory with Th(H_{ω_1})

- Assume $A \subseteq 2^{\mathbb{N}}$ is Σ^1_n. Then A is Σ_{n-1}-definable in H_{ω_1} in the language $\mathcal{L}^*$.
- Assume A is Σ_n-definable in H_{ω_1} in the language $\mathcal{L}^*$. Then $A = \mathrm{Cod}^{-1}[\mathrm{Cod}[A]]$, and $\mathrm{Cod}[A]$ is Σ^1_{n+1}.

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For all n there is some $A_n \in \Sigma^1_{n+1} \setminus \Pi^1_n$. Therefore A_2 is Σ_2-definable but not Π_1-definable in H_{ω_1}.
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Universally Baire sets.

Definition (Feng, Magidor, Woodin)

$A \subseteq 2^{\mathbb{N}}$ is universally Baire if for every compact Hausdorff space X and every continuous $f : X \to 2^{\mathbb{N}}$ we have that $f^{-1}[A]$ has the Baire property in X.

Universal Baireness is a sharp bound between those subsets of $2^{\mathbb{N}}$ which are topologically simple and those which may give troubles and may exist just appealing to the axiom of choice.

Remark

- *All Borel subsets of $2^{\mathbb{N}}$ are universally Baire.*
- *Assuming $V = L$. There are Δ^1_2-subsets of $2^{\mathbb{N}}$ which are not universally Baire.*
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Theorem

Let T be the theory ZFC*+*there are class many Woodin cardinals.*

1. *[1, Thm. 3.3.9, Thm. 3.3.14] Assume V models T. Then every set of reals in $L(\mathbb{R})$ is universally Baire. In particular (since H_{ω_1} is a subset of $L(\mathbb{R})$), all definable subsets of H_{ω_1} with parameters in H_{ω_1} are universally Baire.*
2. *[1, Thm. 3.4.17] Assume V models T, and is obtained as a generic extension of some W such that for some δ which is supercompact in W, we have that $(2^\delta)^W$ is countable in V. Let* UB *be the family of universally Baire sets in V. Then every subset of $2^{\mathbb{N}}$ in $L(\mathrm{Ord}^\omega, \mathrm{UB})^V$ is universally Baire.*

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Model completeness of the theory of H_{ω_1} with universally Baire sets.

Corollary

Let T be the theory ZFC+*there are class many Woodin cardinals. Assume V models T and condition (2) of Thm. 11 holds.*
Let $\mathcal{L}^{**} = \mathcal{L}^{*} \cup \left\{B : \overline{B} \in \mathrm{UB}\right\}$.
Then the $\mathcal{L}^{**}$-theory T_1 of

$$(H_{\omega_1}, \overline{R}_{\phi} : \phi \text{ bounded}, \overline{B} : \overline{B} \in \mathrm{UB})$$

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Let $A \subseteq H_{\omega_1}$ be defined as the extension of some $\mathcal{L}^{**}$-formula $\phi(x, r_1, \ldots, r_n)$ with $r_i \in 2^{\mathbb{N}}$.

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Now

$$A = \{a \in H_{\omega_1} : \forall y(\langle y, a\rangle \in \mathrm{Cod} \to y \in B)\}.$$

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By the third criterion of Robinson's test we conclude that T_1 is model complete. □

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Model completeness of the theory of H_{ω_1} with universally Baire sets.

Corollary

Let T be the theory ZFC+there are class many Woodin cardinals. Assume V models T and condition (2) of Thm. 11 holds. Let:

- *T_0 be the $\mathcal{L}^{**}$-theory of V with parameters in H_{ω_1} and predicates for all elements of UB,*
- *T_1 be the $\mathcal{L}^{**}$-theory with parameters of*

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Then T_1 has quantifier elimination and is the model companion of T_0.

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Proof:

By (a slight variation of the proof of) Levy's absoluteness we have that

$$(H_{\omega_1}, \overline{R}_\phi : \ \phi \text{ bounded}, \overline{B} : \overline{B} \in \text{UB}) \prec_1 (V, \overline{R}_\phi : \ \phi \text{ bounded}, \overline{B} : \overline{B} \in \text{UB}).$$

In particular T_1 and T_0 satisfy the same universal sentences. *(Proof continues)*

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Proof continued:

It is now a standard result in model theory that in this case it is possible to embed any model $\mathcal{M}$ of each theory into some model $\mathcal{N}$ of the other theory by choosing $\mathcal{N}$ saturated enough so to realize all existential types of $\mathcal{M}$ (see for details [2, Lemma 3.1.2]).

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Similarly we get that:

Assume V models that there are class many Woodin cardinals. Then the theory of

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These results are amenable to H_{ω_2} in the presence of forcing axioms. For example I believe (modulo some slight twist) I can prove the following:

Theorem

Assume MM^{+++}. *Let* S_0 *be the* $\mathcal{L}^*$*-theory of* H_{ω_2} *with parameters and* S_1 *be the* $\mathcal{L}^*$*-theory of* V *with parameters in* H_{ω_2}. *Then* S_0 *is the model companion of* S_1.

Proof.

We can still argue (using the generic absoluteness results given by MM^{+++}, some model theoretic trick, and the canonical wel ordering of H_{ω_2} which exists assuming MM) that S_1 is model complete. The rest follows easily as before. □

More generally we will have these type of results whenever we have an axiom granting that the theory of H_κ is generically invariant with respect to forcings preserving the truth of all Δ_1-properties of κ AND this axiom.

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Assume MM^{+++}. Let S_0 be the $\mathcal{L}^*$-theory of H_{ω_2} with parameters and S_1 be the $\mathcal{L}^*$-theory of V with parameters in H_{ω_2}. Then S_0 is the model companion of S_1.

Proof.

We can still argue (using the generic absoluteness results given by MM^{+++}, some model theoretic trick, and the canonical wel ordering of H_{ω_2} which exists assuming MM) that S_1 is model complete. The rest follows easily as before. □

More generally we will have these type of results whenever we have an axiom granting that the theory of H_κ is generically invariant with respect to forcings preserving the truth of all Δ_1-properties of κ AND this axiom.

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THANKS FOR WATCHING!

Cohen's absoluteness Lemma

Lemma

Assume that:

- $\phi(x, r)$ *is a* Δ_0*-formula with real parameter* r.
- $\mathsf{B} \in V$ *is a Boolean algebra such that*

$$V \models (1_{\mathsf{B}} = [\![\exists x \phi(x, \check{r})]\!]_{\mathsf{B}}).$$

Then $H_{\omega_1} \models \exists x \phi(x, r)$.

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Assume B $\in V$ is a complete boolean algebra such that

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To simplify matters assume there is an inaccessible λ such that B $\in V_\lambda$ (redundant assumption).
Then $V_\lambda \models$ ZFC and

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Pick $N \prec V_\lambda$ countable such that B $\in N$.
Let $M = \pi_N[N]$ (where π_N is the Mostowski collapsing map of N), and Q $= \pi_N$(B). Notice that $r \in \mathcal{P}(\mathbb{N})$ and $\pi_N(\mathbb{N}) = \mathbb{N}$ (since $\mathbb{N} = \omega \subseteq N$), Thus $\pi_N(r) = r$.

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Since $\pi_N : N \to M$ is an isomorphism and $\mathbb{Q} = \pi_N(\mathbb{B})$, $\pi_N(r) = r$,

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Now M is *countable* and transitive, $\mathbb{Q} \in M$, hence we can find in V an M-generic filter $G \in V$ for $\mathbb{Q}$.

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By Cohen's forcing Theorem we can define $\sigma_G : M^{\mathbb{Q}} \to M[G]$ surjective such that

- $\sigma_G(\check{a}) = a$ for all $a \in M$,
- $M[G]$ is transitive,
- $M[G] \models \psi$ iff $[\![\psi]\!]_{\mathbb{Q}} \in G$.

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Thus there is $a \in M[G]$ such that $M[G] \models \phi(a, r)$.
Since $M[G]$ is countable and transitive, $M[G] \in H_{\omega_1}$ and $M[G] \subseteq H_{\omega_1}$, thus $a, r \in H_{\omega_1}$.
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Cohen's absoluteness Lemma reformulated

Actually if one doesn't want to commit to any philosophical position on the onthology of sets Cohen's absoluteness Lemma can be formulated as follows:

Corollary (Cohen)

Let T be any first order theory which extends ZFC and $\phi(x, r)$ be a Σ_0 formula with a parameter r such that $T \vdash r \subseteq \omega$. TFAE:

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Any model of $\Omega(V)$ embeds into one of the form $H^{\mathsf{B}}_{\check{\omega}_1}/_G$.

Let $H^{\mathsf{B}}_{\kappa}/_G \in \Omega(V)$. Find a regular $\delta > 2^\kappa$ and consider the forcing notion $\text{Coll}(\omega, < \delta)$.

By a classical forcing result, we have that B is isomorphic to a complete subalgebra of the boolean completion C of $\text{Coll}(\omega, < \delta)$.

W.l.o.g. we can assume that C contains B as a complete subalgebra.

Extend $G \in \text{St}(\mathsf{B})$ to an ultrafilter H on C.

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Back to Model companions in the generic multiverse

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Recall on boolean algebras and Stone spaces

Given a boolean algebra B:

- St(B) is given by its ultrafilters G.
- St(B) is endowed with a *compact, Hausdorff* topology τ_B whose clopens are $N_b = \{G \in \mathrm{St(B)} : b \in G\}$.
- The map $b \mapsto N_b$ defines a natural isomorphism of B with the boolean algebra CLOP(St(B)) of clopen subset of St(B).
- B is *complete* if and only if $\mathrm{CLOP(St(B))} = \mathrm{RO(St(B)}, \tau_B) \cong \mathrm{B}$.
- Spaces X satisfying the property that its regular open sets are closed are *extremally (or extremely) disconnected.*
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Boolean valued models

Definition

Let B be a *cba* and a $\mathcal{L}$ be first order *relational* language.
A B-*valued model* for $\mathcal{L}$ is a tuple
$\mathcal{M} = \langle M, =^{\mathcal{M}}, R_i^{\mathcal{M}} : i \in I, c_j^{\mathcal{M}} : j \in J \rangle$ with

$$\begin{aligned} =^{\mathcal{M}} &: M^2 \to \mathsf{B} \\ (\tau, \sigma) &\mapsto [\![\tau = \sigma]\!]^{\mathcal{M}}_{\mathsf{B}} = [\![\tau = \sigma]\!], \end{aligned}$$

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Forcing relations on boolean valued models

The constraints on $R^{\mathcal{M}}$ and $=^{\mathcal{M}}$ are the following:

- for $\tau, \sigma, \chi \in M$,
  1. $[\![\tau = \tau]\!] = 1_{\mathsf{B}}$;
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Boolean valued semantics

Definition

Let $\langle M, =^{\mathcal{M}}, R^{\mathcal{M}} \rangle$ be a B-valued model in the relational language $\mathcal{L} = \{R\}$, $\phi(x_1, \ldots, x_n)$ a $\mathcal{L}$-formula with displayed free variables, ν : free variables $\to M$.

$[\![\phi]\!]_{\mathsf{B}}^{\mathcal{M},\nu} = [\![\phi]\!]$, the *boolean value* of ϕ with the assignment ν is defined by recursion as follows:

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Soundness Theorem for B-valued semantics

Theorem (Soundness Theorem)

Assume $\mathcal{L}$ is a relational language and ϕ is a $\mathcal{L}$-formula which is syntactically provable by a $\mathcal{L}$-theory T.
Assume each formula in T has boolean value at least $b \in \mathsf{B}$ in a B-valued model $\mathcal{M}$ with valuation ν.
Then $[\![\phi]\!]_{\mathsf{B}}^{\mathcal{M},\nu} \geq b$ as well.

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Tarski quotient of B-valued models

Definition

Let B be a *cba*, $\mathcal{M}$ a B-valued model for $\mathcal{L}$, and G a ultrafilter over B. Consider the following equivalence relation on M:

$$\tau \equiv_G \sigma \Leftrightarrow [\![\tau = \sigma]\!] \in G$$

The first order (Tarski) model $\mathcal{M}/_G = \langle M/_G, R_i^{\mathcal{M}/_G} : i \in I, c_j^{\mathcal{M}/G} : j \in J \rangle$ is defined letting:

- For any n-ary relation symbol R in $\mathcal{L}$

$$R^{\mathcal{M}/G} = \{([\tau_1]_G, \ldots, [\tau_n]_G) \in (M/_G)^n : [\![R(\tau_1, \ldots, \tau_n)]\!] \in G\}.$$

- For any constant symbol c in $\mathcal{L}$

$$c^{\mathcal{M}/G} = [c^{\mathcal{M}}]_G.$$

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Full B-valued models

Definition

A B-valued model $\mathcal{M}$ for the language $\mathcal{L}$ is *full* if for every $\mathcal{L}$-formula $\phi(x, \bar{y})$ and every $\bar{\tau} \in M^{|\bar{y}|}$ there is a $\sigma \in M$ such that

$$[\![\exists x \phi(x, \bar{\tau})]\!] = [\![\phi(\sigma, \bar{\tau})]\!]$$

Boolean valued Łoś Theorem — Forcing theorem

Theorem (B-valued Łoś's Theorem — Forcing theorem)

Assume $\mathcal{M}$ is a full B*-valued model for the relational language $\mathcal{L}$. Then for every formula $\phi(x_1, \ldots, x_n)$ in $\mathcal{L}$ and $(\tau_1, \ldots, \tau_n) \in M^n$:*

1. *For all ultrafilters G over* B*, $\mathcal{M}/G \models \phi([\tau_1]_G, \ldots, [\tau_n]_G)$ if and only if $[\![\phi(\tau_1, \ldots, \tau_n)]\!] \in G$.*
2. *For all $a \in$ B the following are equivalent:*
    1. $[\![\phi(f_1, \ldots, f_n)]\!] \geq a$,
    2. $\mathcal{M}/G \models \phi([\tau_1]_G, \ldots, [\tau_n]_G)$ *for all $G \in N_a$,*
    3. $\mathcal{M}/G \models \phi([\tau_1]_G, \ldots, [\tau_n]_G)$ *for densely many $G \in N_a$.*

Łoś's Theorem versus boolean valued Łoś's Theorem

Fact

Let $(M_x : x \in X)$ be a family of Tarski-models in the first order relational language $\mathcal{L}$. Then $N = \prod_{x \in X} M_x$ is a full $\mathcal{P}(X)$-model, letting for each n-ary relation symbol $R \in \mathcal{L}$,
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Let G be any non-principal ultrafilter on X. Then the Tarski quotient N/G is the familiar ultraproduct of the family $(M_x : x \in X)$ by G.

The usual Łoś theorem for ultraproducts of Tarski models is the specialization to the case of the full $\mathcal{P}(X)$-valued model N of the boolean valued Łoś theorem.

If N is an ultrapower of a model M, the embedding $a \mapsto [c_a]_G$ (where $c_a(x) = a$ for all $x \in X$ and $a \in M$) is elementary.

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$$V^{\mathsf{B}} = \{\tau : \tau : V^{\mathsf{B}} \to \mathsf{B}$$

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Moreover V^{B} models ZFC, and for κ regular in V^{B}, H_κ^{B} models ZFC except (may be) the power-set axiom.

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