Conciliatory Sets and Unambiguous Tree Automata

Jacques Duparc

UNIL | Université de Lausanne

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

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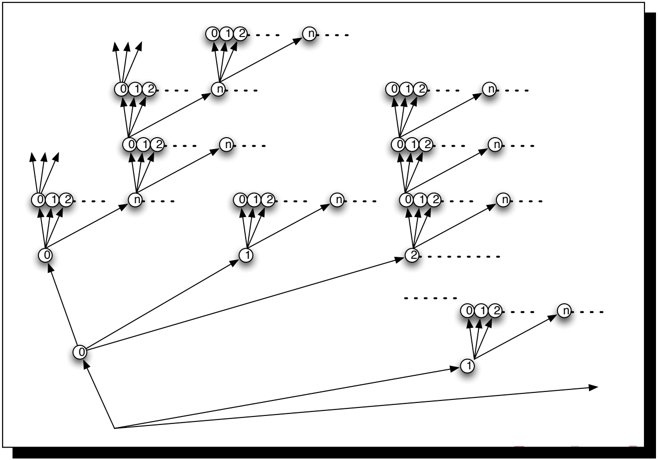
Joint work with

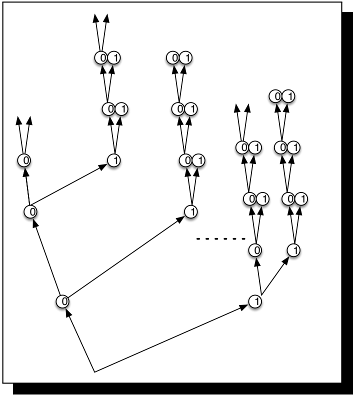
- Kevin Fournier Kevin.Fournier@unil.ch
- Szczepan Hummel shummel@mimuw.edu.pl

- $\mathcal{N}$ = Baire Space = set of infinite sequences of integers = $\mathbb{N}^{\mathbb{N}} = \mathbb{N}^{\omega}$

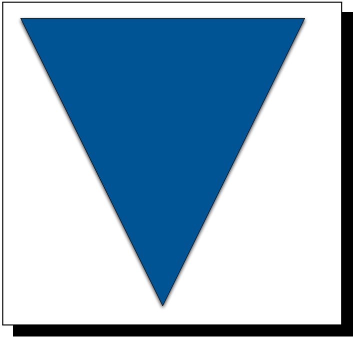
$$\{f : \; f \text{ mapping} : \; \mathbb{N} \longmapsto \mathbb{N}\}$$



- Cantor Space $= \{0,1\}^{\mathbb{N}} = 2^{\omega}$



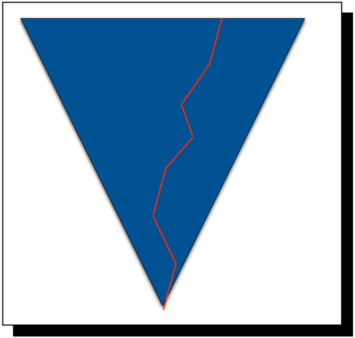
Background



For simplicity:

$$\mathcal{N}$$

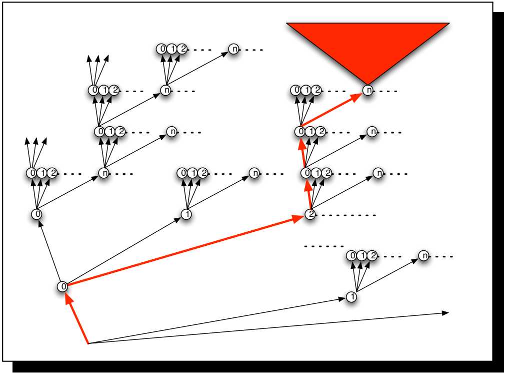
Background



A point = an infinite sequence inside $\mathcal{N}$

Background

- Product topology of the discrete topology on $\mathbb{N}$.
- Basic open set: $u\mathcal{N}$ where u is a finite sequence of integers



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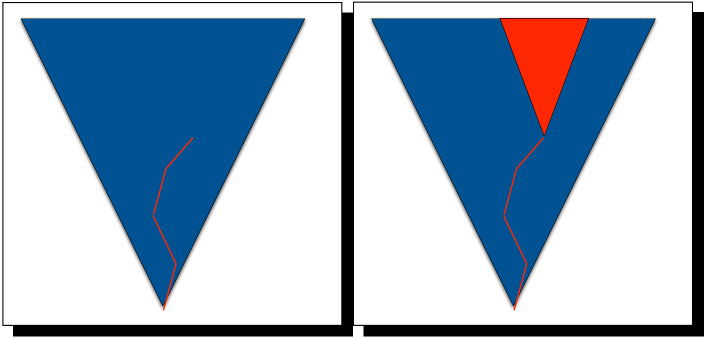

A finite sequence yields a basic open set of $\mathcal{N}$

Background

- $\mathcal{N}$ is separable (has a countable dense subset) since it is second countable (has a countable basis)

$$\{u\mathcal{N} : \ u \in \mathbb{N}^{<\omega}\}$$

- $\mathcal{N}$ is metrizable = there exists a metric compatible with the topology

$$dist(x, y) = 2^{-n} \text{ where } n \text{ is first s.t. x(n)} \neq \text{y(n)}$$

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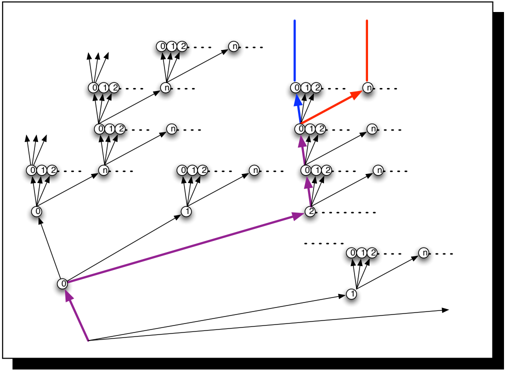
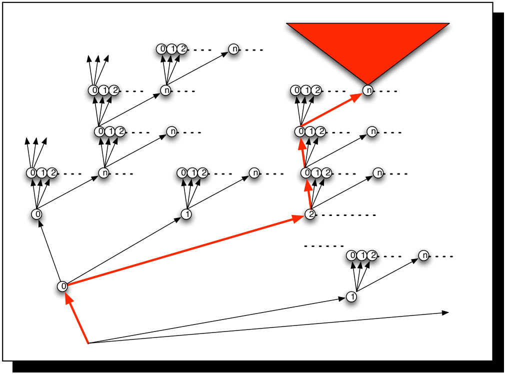


$$dist(x,y) = 2^{-4}$$

Background



A finite sequence yields a basic open set of $\mathcal{N}$

open ball $B(0200n\ldots, 2^{-5})$

Background

- Open set: $W\mathcal{N}$ where W is a set of finite sequence of integers.

- Closed set $(W\mathcal{N})^{\complement}$

$$T = \{v \in \mathbb{N}^{<\omega} : \ \forall u \text{ prefix of } v, \ u \notin W\},$$

T is a tree, and $x \in (W\mathcal{N})^{\complement} \iff x$ is an infinite branch of T.

- X is closed $\iff$ X is the set of infinite branches of a tree.

Background

- $\mathcal{N}$ is homeomorphic to $(\mathbb{R} \setminus \mathbb{Q})$ with the induced topology.
- Usual proof uses continued fractions.
- Here we present a direct proof (Miller)

Background

- $\mathcal{N}$ is homeomorphic to $(\mathbb{R} \setminus \mathbb{Q})$ with the induced topology.
- First replace $\mathbb{N}$ by $\mathbb{Z}$.
- There is an obvious homeomorphism between $\mathbb{Z}^\omega$ and $\mathbb{N}^\omega$:

$$\phi(f) = g$$

  - $f(n) = g(2n)$ if $n \in \mathbb{N}$
  - $f(-n) = g(2n-1)$ if $-n \in \mathbb{Z} \smallsetminus \mathbb{N}$

Background

- Construct a mapping from by $\mathbb{Z}^\omega$ to $(\mathbb{R} \setminus \mathbb{Q})$.
- Enumerate the rationale

$$\mathbb{Q} = \{q_n : \; n \in \mathbb{N}\}$$

- Inductively define a sequence of open intervals

$$\left\langle I_s : \; s \in \mathbb{Z}^{<\omega} \right\rangle$$

1. $I_{\langle\rangle} = \mathbb{R}$, and for $s \neq \langle\rangle$, each I_s is a nontrivial open interval in $\mathbb{R}$ with rational endpoints,
2. for every $s \in \mathbb{Z}^{<\omega}$, for every $a \in \mathbb{Z}$, $\; I_{s\frown a} \subset I_s$,
3. the right end point of $I_{s\frown a}$ is the left end point of $I_{s\frown a+1}$,
4. $\{I_{s\frown a} : \; a \in \mathbb{Z}\}$ covers all of I_s except for their endpoints,
5. the length of I_s is less than $\frac{1}{|s|}$,
6. the the n$^{\text{th}}$ rational q_n is an endpoint of I_t for some $|t| < n + 1$,

Background

- Construct a mapping $\phi : \mathbb{Z}^\omega \longmapsto (\mathbb{R} \setminus \mathbb{Q})$ as follows.
- Given any $x \in \mathbb{Z}^\omega$, the set

$$\bigcap_{n \in \mathbb{N}} I_{x \restriction n}$$

  is a singleton that contains an irrational,
  - It is non empty because

$$closure(I_{x \restriction n+1}) \subseteq I_{x \restriction n},$$

  - It is a singleton because

$$\lim_{n \to \infty} diam(I_{x \restriction n}) = 0,$$

- ϕ is defined by

$$\{\phi(x)\} = \bigcap_{n \in \mathbb{N}} I_{x \restriction n}$$

Background

- The function $\phi : \mathbb{Z}^\omega \longmapsto (\mathbb{R} \setminus \mathbb{Q})$ is
  - one-to-one because if s and t are incomparable then I_s and I_t are disjoint.
  - It is onto since for every $y \in (\mathbb{R} \setminus \mathbb{Q})$ and every $n \in \mathbb{N}$ there is a unique s of length n with $y \in I_s$.
- It is a homeomorphism because
  - $$\phi[s\mathcal{N}] = I_s \cap (\mathbb{R} \setminus \mathbb{Q})$$
  - and the sets of the form $I_s \cap (\mathbb{R} \setminus \mathbb{Q})$ form a basis for $(\mathbb{R} \setminus \mathbb{Q})$.

The Borel Hierarchy

The Borel hierarchy

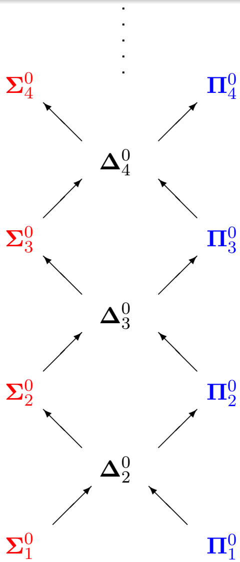
Definition

1. $\mathbf{\Sigma}^0_1 = \{open\ sets\}$
2. $\mathbf{\Pi}^0_\alpha = \{A^\complement : \ A \in \mathbf{\Sigma}^0_\alpha\}$
3. $\mathbf{\Delta}^0_\alpha = \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha$
4. $\mathbf{\Sigma}^0_\alpha = \{A = \bigcup_{n\in\mathbb{N}} A_n : \ A_n \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta\}$

$$\mathcal{B} = \bigcup_{\alpha\in On} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha\in On} \mathbf{\Pi}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha$$

In particular, $\mathbf{\Pi}^0_1$ is the class of all closed sets, $\mathbf{\Sigma}^0_2$ the class of all countable union of closed sets, etc...

The Borel hierarchy



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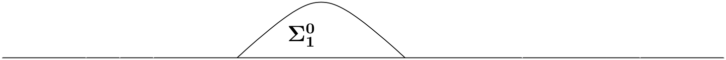

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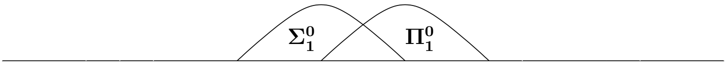

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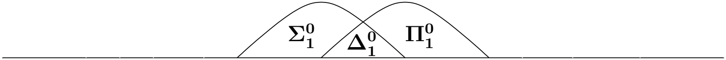

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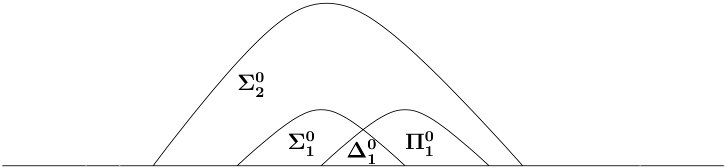

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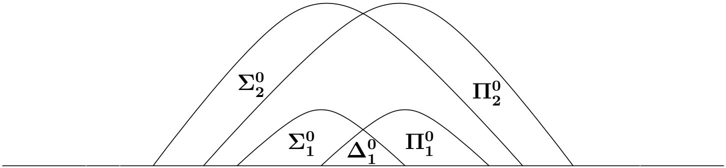

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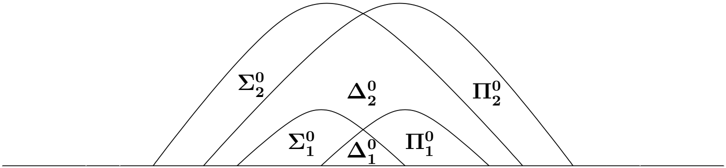

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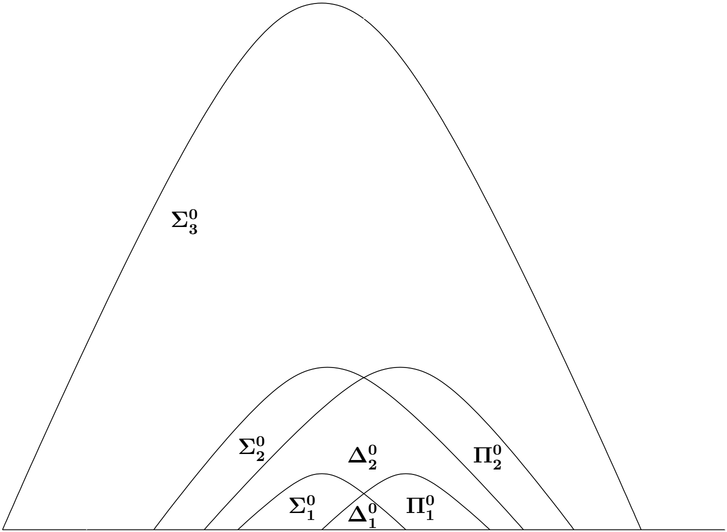

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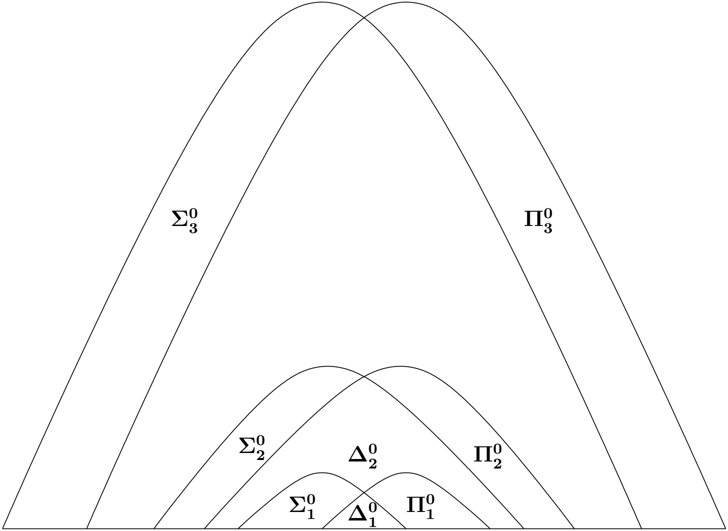

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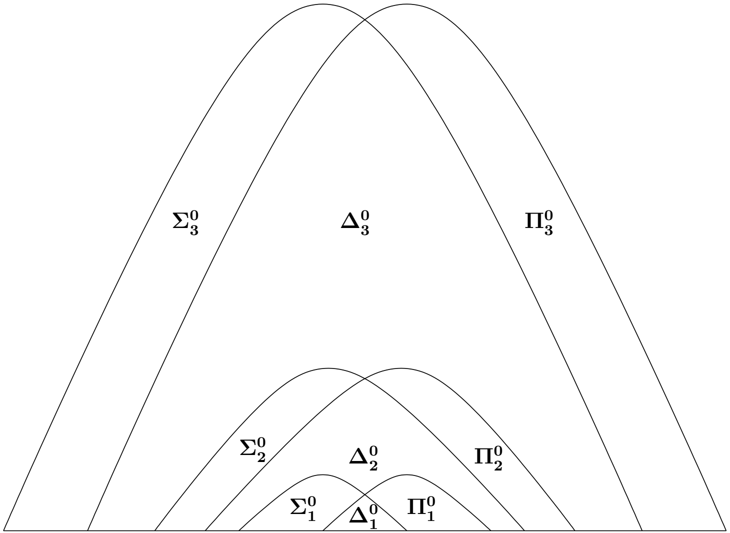
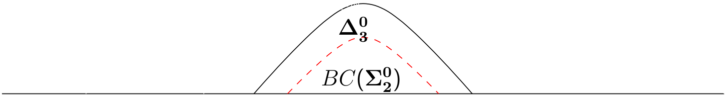
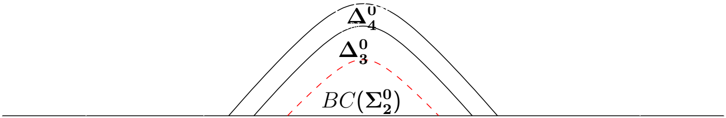



$\mathbf{\Delta^0_3}$
$BC(\mathbf{\Sigma^0_2})$

The Borel hierarchy



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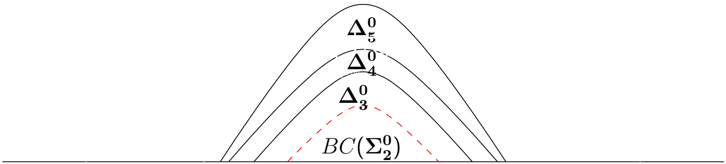

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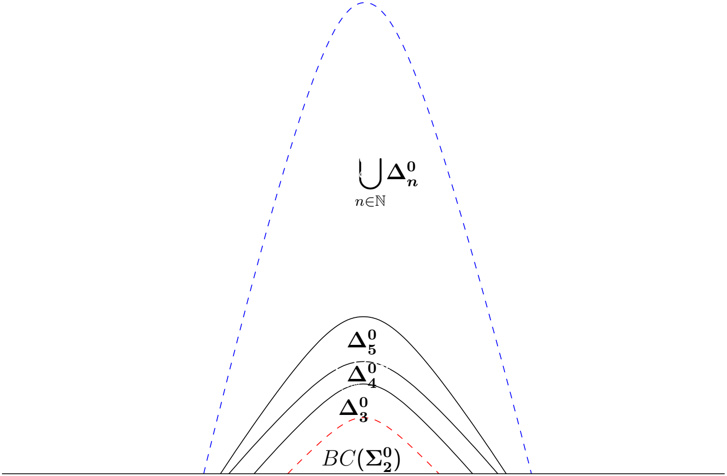

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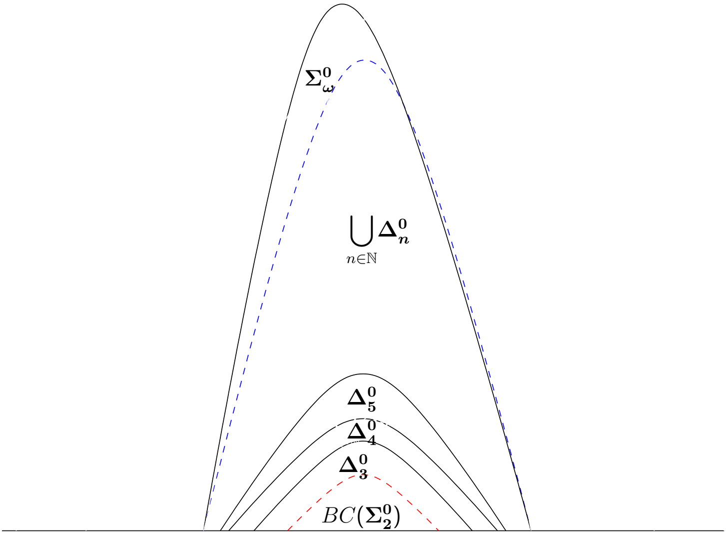

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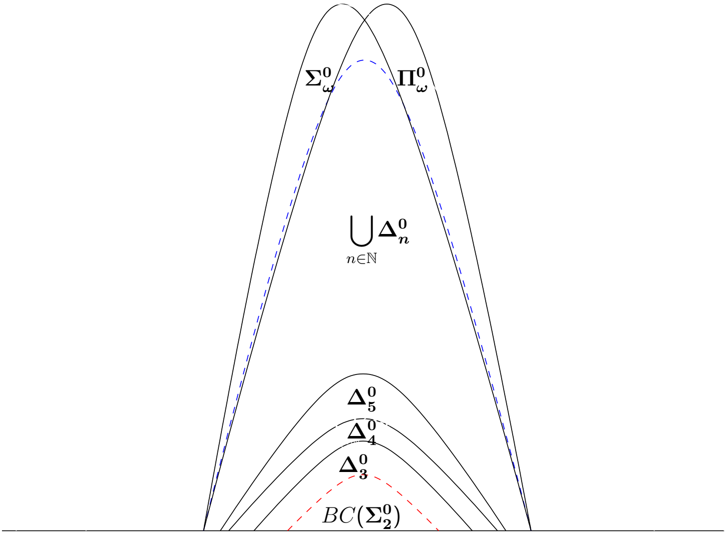

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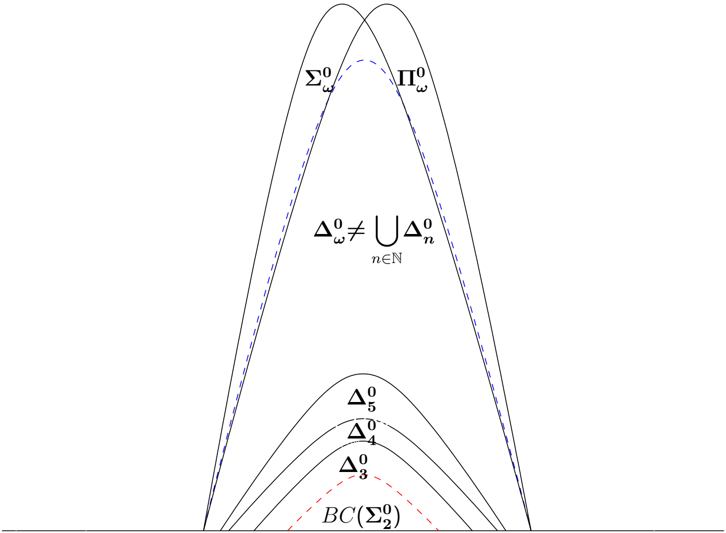

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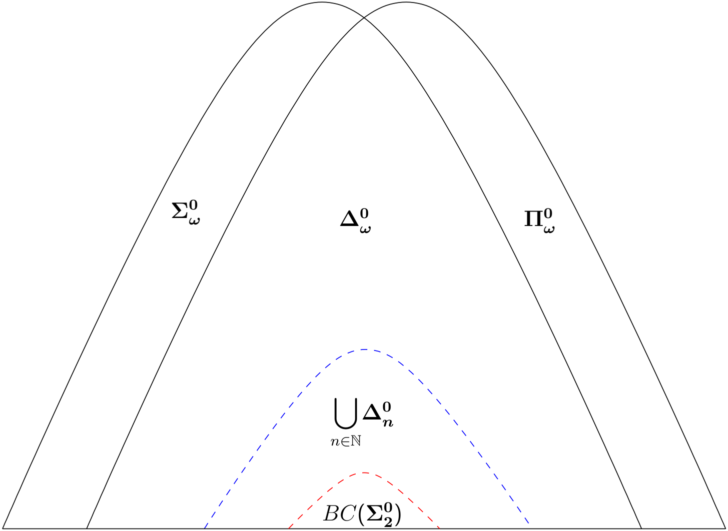

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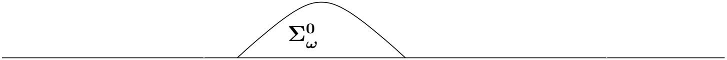
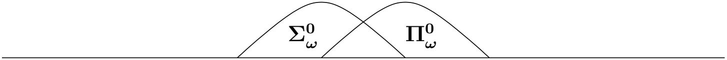
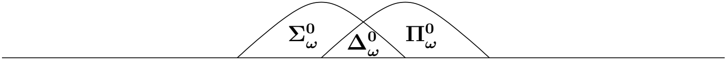



$\mathbf{\Sigma}^0_\omega$
$\mathbf{\Pi}^0_\omega$

The Borel hierarchy



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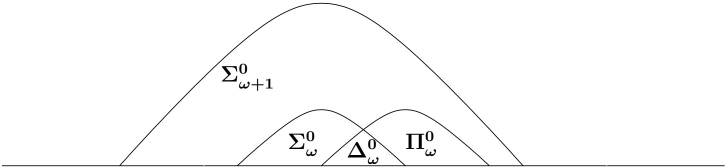

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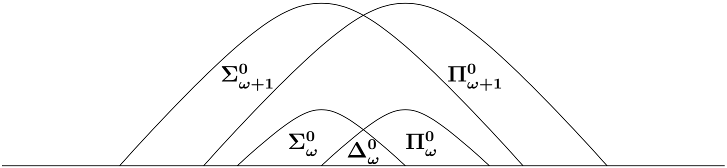

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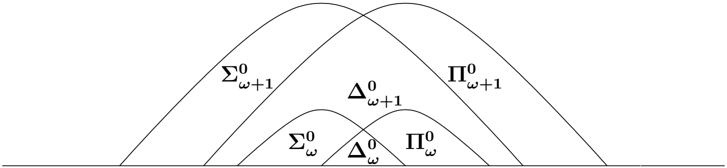

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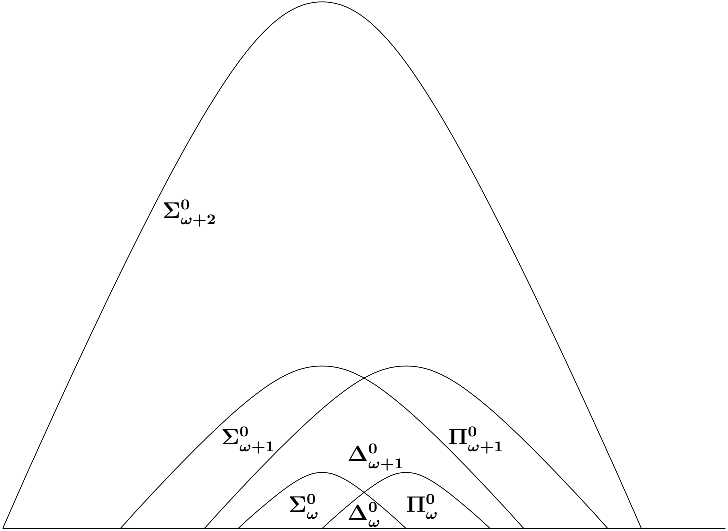

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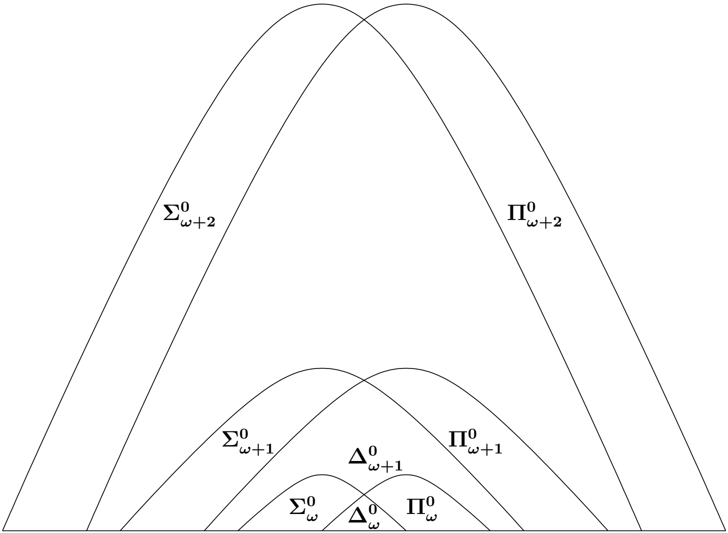

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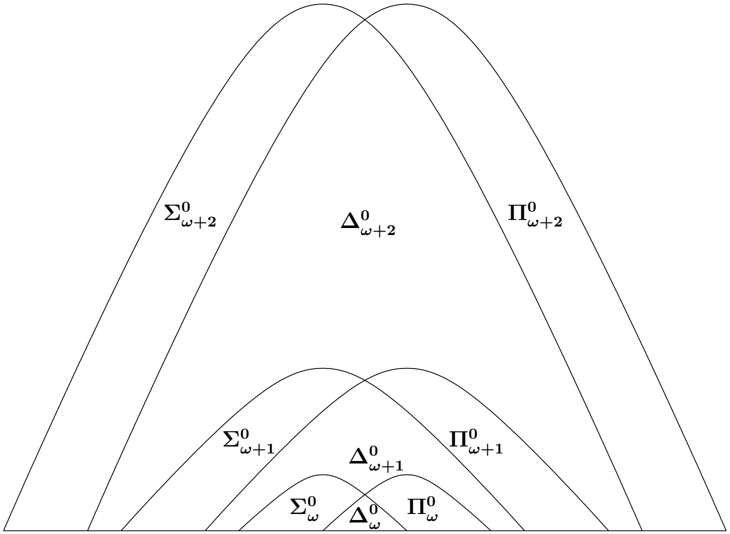


The Projective Hierarchy

Definition

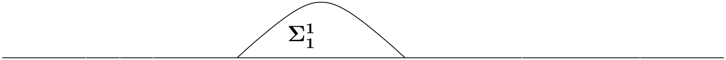
$A \subseteq S$ is analytic $\iff$ $x \in A \Leftrightarrow \exists y\ (x, y) \in B$, $B \in S \times \mathbb{N}^\omega$, B closed (equivalently Borel)

1. $\mathbf{\Sigma^1_1} = \Big\{ \text{Analytic sets} \Big\}$
2. $\mathbf{\Pi^1_n} = \Big\{ S \setminus A : A \in \mathbf{\Sigma^1_n} \Big\}$
3. $\mathbf{\Delta^1_n} = \mathbf{\Sigma^1_n} \cap \mathbf{\Pi^1_n}$
4. $\mathbf{\Sigma^1_{n+1}} = \Big\{ A \subseteq S : x \in A \Leftrightarrow \exists y\ (x, y) \in B \in \mathbf{\Pi^1_n}(S \times \mathbb{N}^\omega) \Big\}$

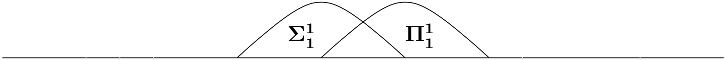
Theorem (Suslin)

$$Borel = \Delta^1_1$$

The Projective hierarchy



The Projective hierarchy



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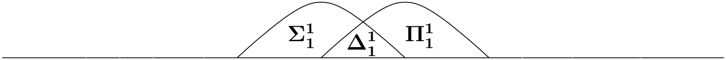

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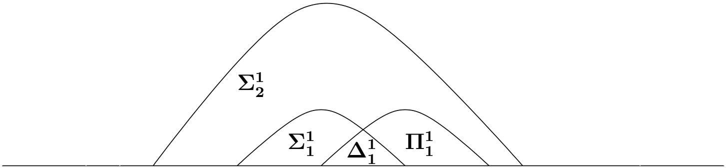
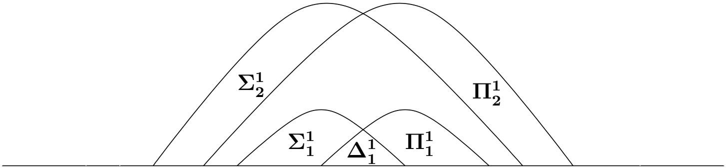
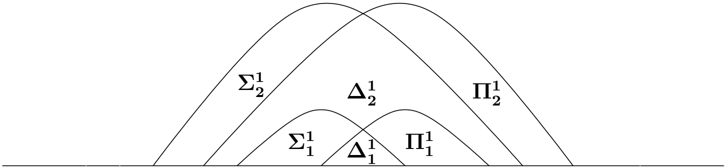




$\mathbf{\Sigma}^1_2$
$\mathbf{\Pi}^1_2$
$\mathbf{\Sigma}^1_1$
$\mathbf{\Delta}^1_1$
$\mathbf{\Pi}^1_1$

The Projective hierarchy



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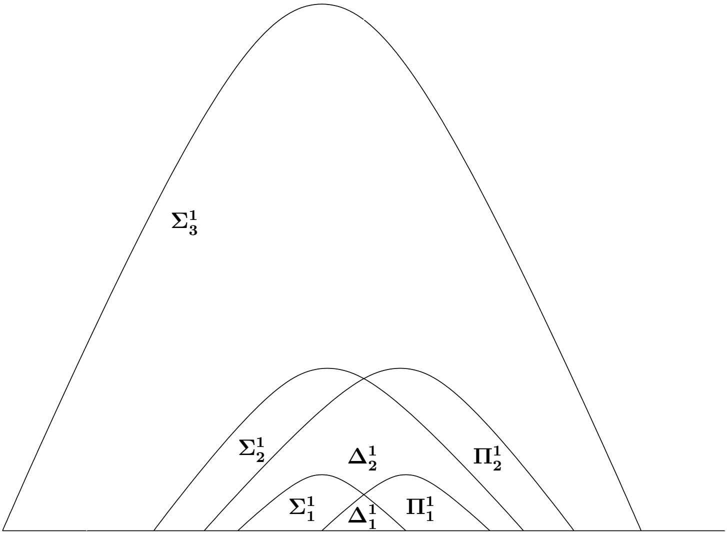

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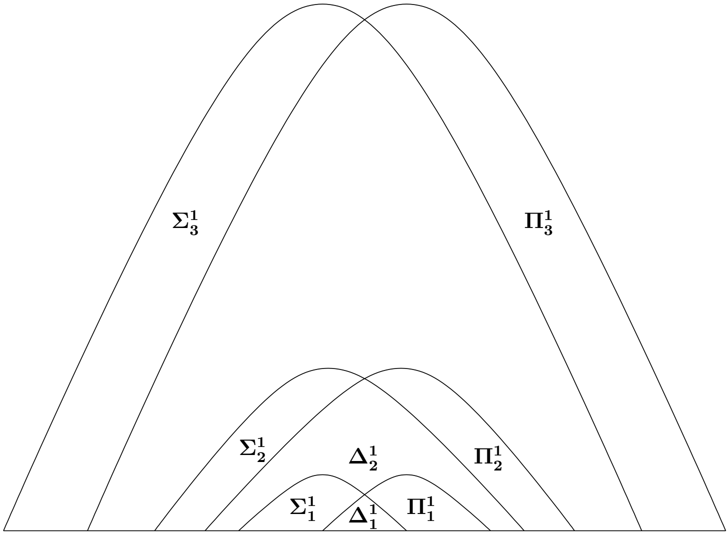

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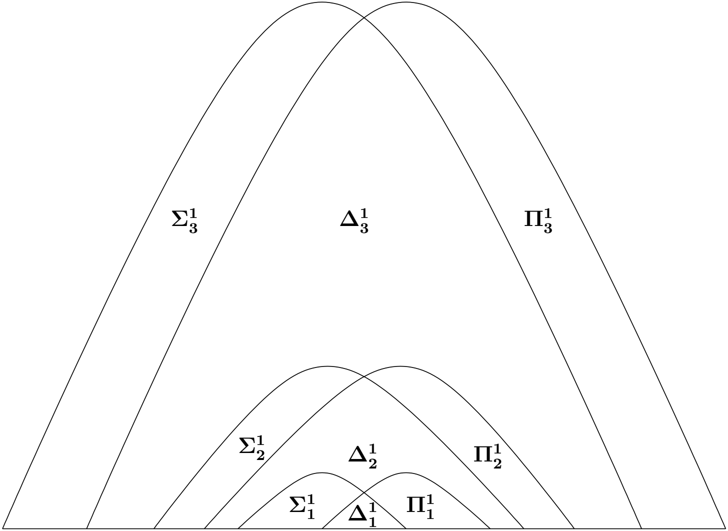
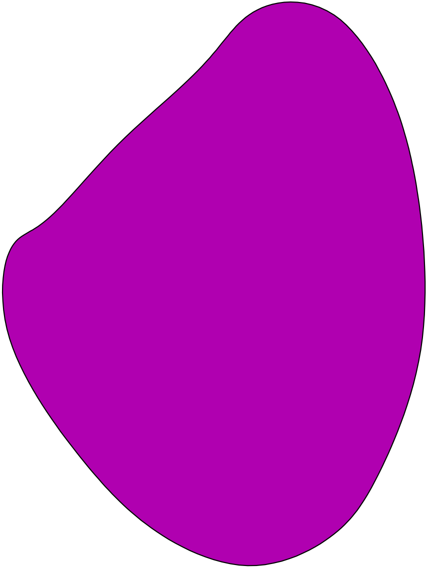


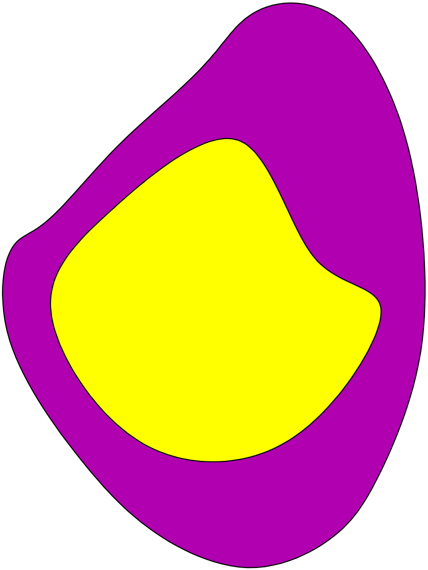
Kuratowski Hierarchy

$\mathrm{Diff}_1(\boldsymbol{\Sigma}^0_3)$

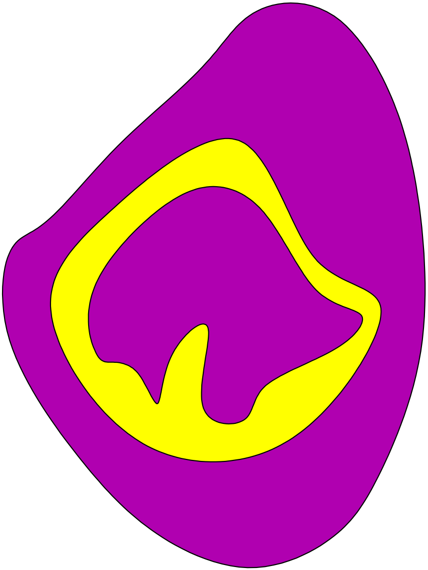


Kuratowski Hierarchy

$\mathrm{Diff}_2(\boldsymbol{\Sigma}^0_3)$

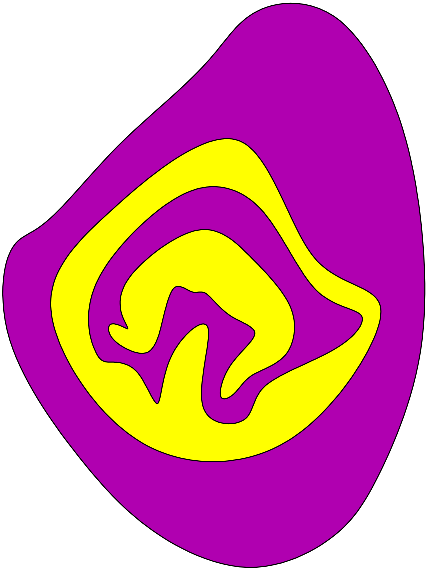


$\mathrm{Diff}_3(\boldsymbol{\Sigma}^0_3)$



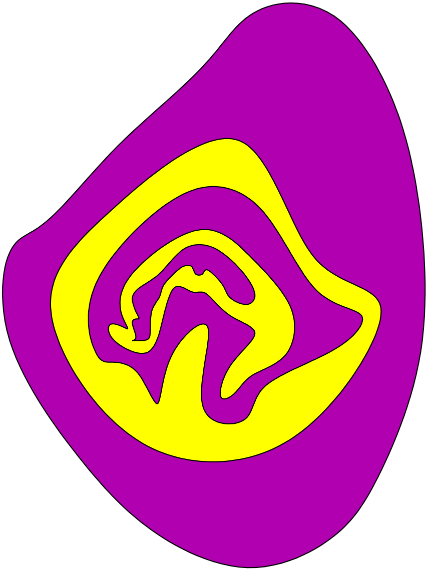
Kuratowski Hierarchy

$\mathrm{Diff}_4(\boldsymbol{\Sigma}^0_3)$



Kuratowski Hierarchy

$\mathrm{Diff}_5(\boldsymbol{\Sigma}^0_3)$



Theorem (Hausdorff-Kuratowski)

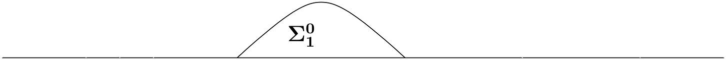
$$\bigcup_{\xi<\omega_1} \mathrm{Diff}_{\xi}(\mathbf{\Sigma}^0_{\alpha}) = \mathbf{\Delta}^0_{\alpha+1}$$

How many levels on Borel sets:

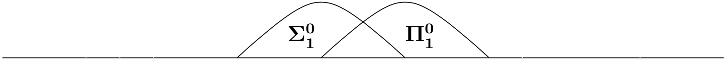
- Borel hierarchy: ω_1
- Hausdorff-Kuratowski hierarchy: $\omega_1 \cdot \omega_1 = \omega_1^2$
- Wadge hierarchy: first fixed point of the first ω_1-many Veblen functions

1. $V_0(\alpha) = \omega_1^{\alpha}$
2. V_{ξ} enumerates the ordinals that are fixed points of all $(V_{\zeta})_{\zeta<\xi}$

Wadge Hierarchy



Wadge Hierarchy



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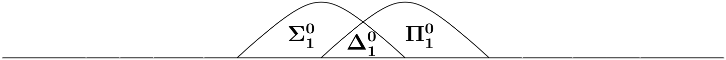
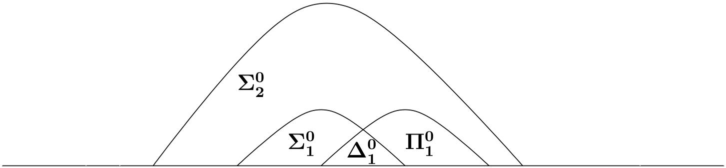
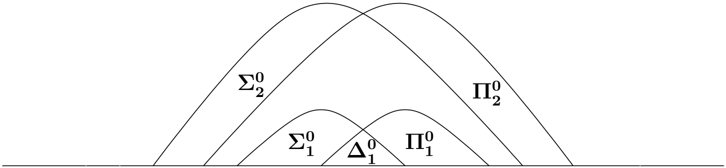


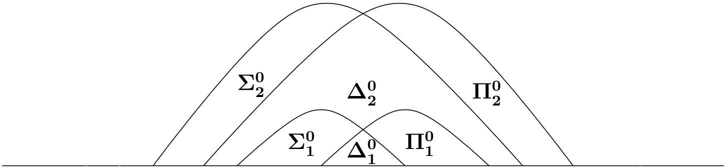
Wadge Hierarchy



Wadge Hierarchy

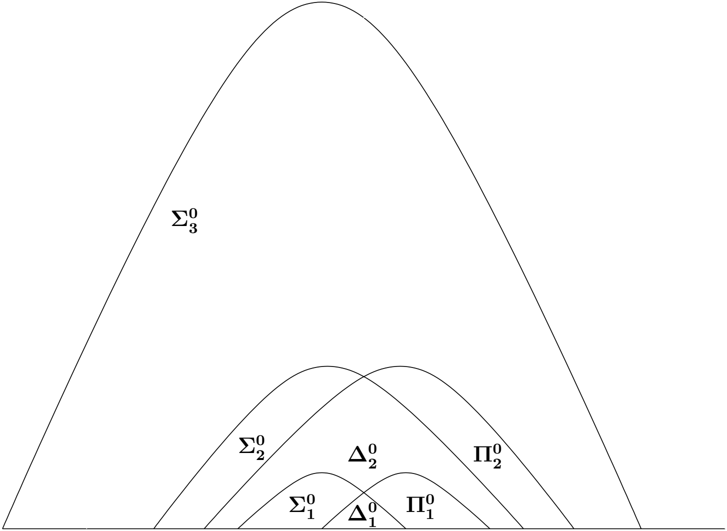


ω_1

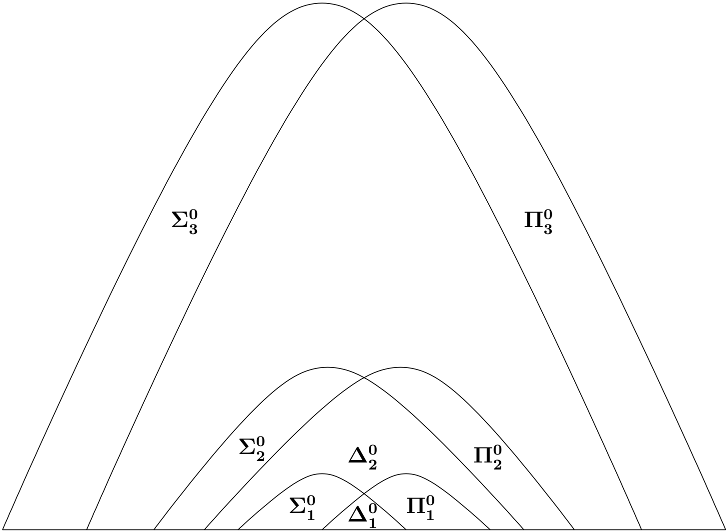


$\mathbf{\Sigma}^0_2$
$\mathbf{\Delta}^0_2$
$\mathbf{\Pi}^0_2$
$\mathbf{\Sigma}^0_1$
$\mathbf{\Delta}^0_1$
$\mathbf{\Pi}^0_1$

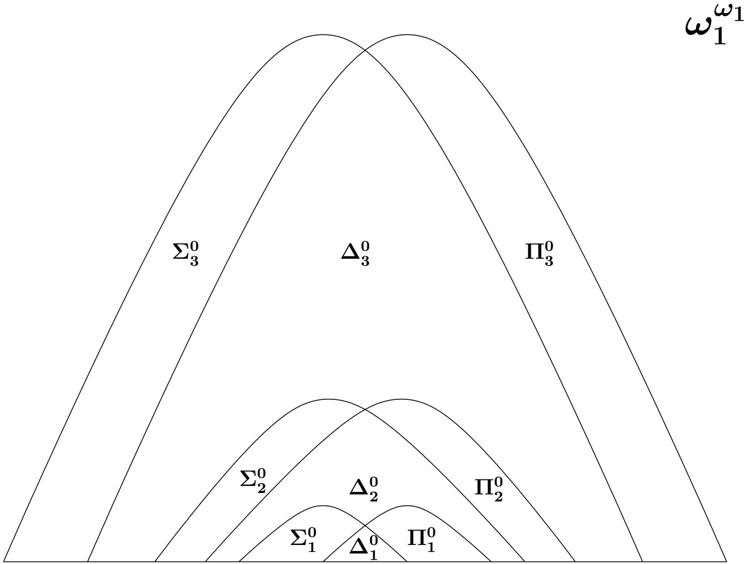
Wadge Hierarchy

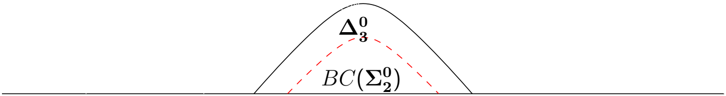


Wadge Hierarchy

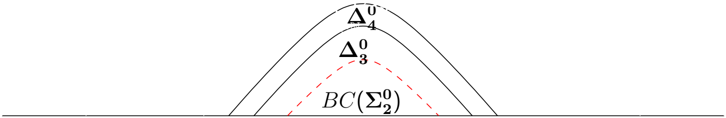


Wadge Hierarchy

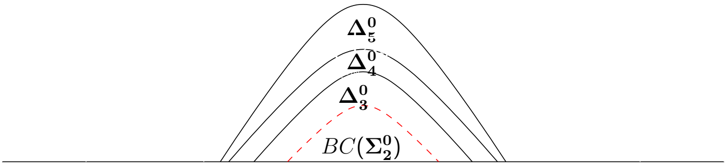



$\boldsymbol{\Delta}^0_3$
$BC(\boldsymbol{\Sigma}^0_2)$

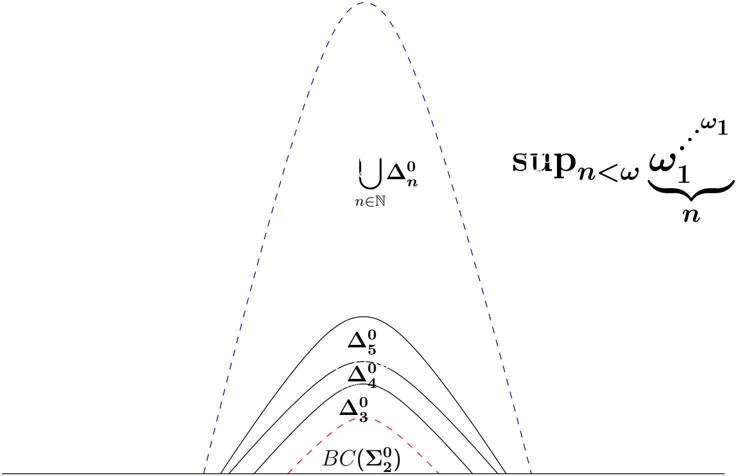
Wadge Hierarchy



Wadge Hierarchy

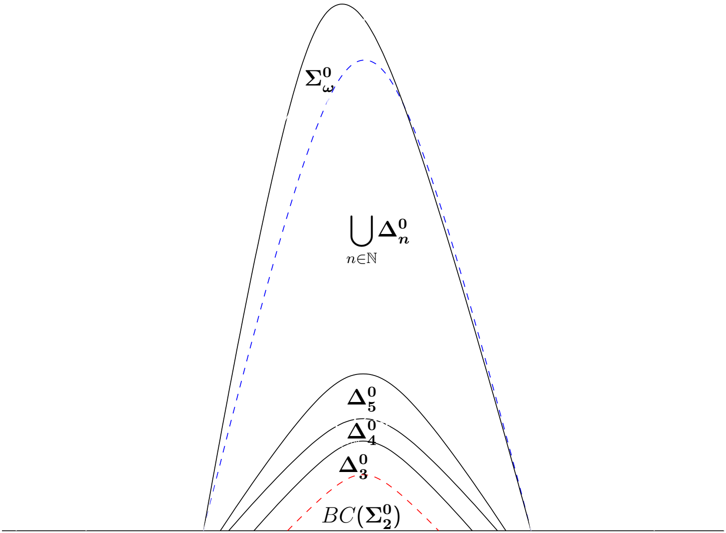


Wadge Hierarchy

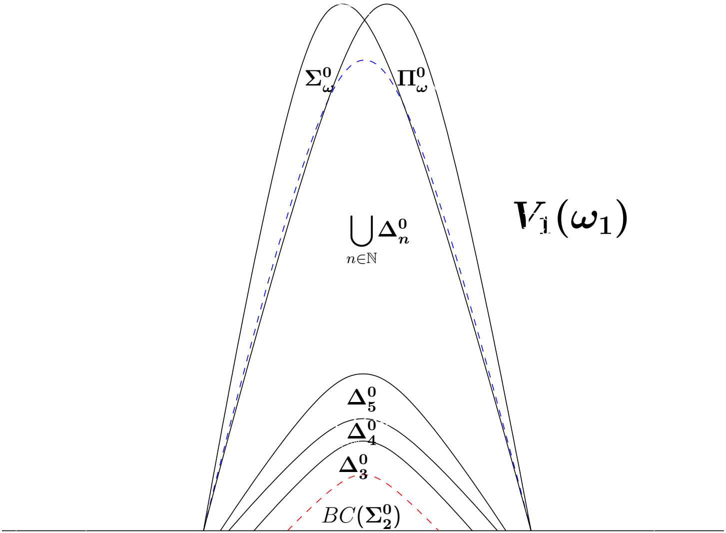


$$\sup_{n<\omega} \underbrace{\omega_1^{\cdot^{\cdot^{\cdot^{\omega_1}}}}}_{n}$$

Wadge Hierarchy



Wadge Hierarchy



Functions as Strategies

- **Wadge hierarchy**: reductions by continuous functions.
- Design **rules** for $\mathbb{G}^{\{R_1,R_2,\dots\}}(f)$ s.t.

$$\textit{II} \text{ has a w.s.} \iff f \text{ is continuous}$$

- f is **continuous** $\iff f^{-1}\mathbf{\Sigma}^0_1 = \mathbf{\Sigma}^0_1$
  $\iff f^{-1}\mathbf{\Pi}^0_1 = \mathbf{\Pi}^0_1$
  $\iff f^{-1}\mathbf{\Delta}^0_1 = \mathbf{\Delta}^0_1$
- $\mathbf{\Delta}^0_1$ means *"I know for sure after k steps"*

Functions as Strategies

I $\quad G^{\{\}}_{(f)} \quad$ II

II wins iff $f(x) = $ y

Functions as Strategies

I $\quad G^{\{\}}_{(f)} \quad$ II

x_0

II wins iff $f(x) = \mathrm{y}$

Functions as Strategies

I $\quad G^{\{\}}_{(f)} \quad$ II

x_0

$\searrow$

II wins iff $f(x) =$ y

Functions as Strategies

I $G^{\{\}}_{(f)}$ II

x_0

$\searrow$

y_0

II wins iff $f(x) = y$

Functions as Strategies

I	$G_{(f)}^{\{\}}$	II
x_0		
	$\searrow$	
		y_0
	$\swarrow$	

II wins iff $f(x) = \mathsf{y}$

Functions as Strategies

I	$G^{\{\}}_{(f)}$	II
x_0		
	$\searrow$	
		y_0
	$\swarrow$	
x_1		

II wins iff $f(x) = $ y

Functions as Strategies

$\mathbf{I}$	$G^{\{\}}_{(f)}$	$\mathbf{II}$
x_0		
	$\searrow$	
		y_0
	$\swarrow$	
x_1		
	$\searrow$	

II wins iff $f(x) = \mathsf{y}$

Functions as Strategies

$$\begin{array}{ccc} \text{I} & G^{\{\}}_{(f)} & \text{II} \\ x_0 & & \\ & \searrow & \\ & & y_0 \\ & \swarrow & \\ x_1 & & \\ & \searrow & \\ & & y_1 \end{array}$$

II wins iff $f(x) =$ y

Functions as Strategies

I	$G^{\{\}}_{(f)}$	II
x_0		
	$\searrow$	
		y_0
	$\swarrow$	
x_1		
	$\searrow$	
		y_1
	$\swarrow$	

II wins iff $f(x) = $ y

Functions as Strategies

I	$G^{\{\}}_{(f)}$	II
x_0		
	$\searrow$	
		y_0
	$\swarrow$	
x_1		
	$\searrow$	
		y_1
	$\swarrow$	
x_2		

II wins iff $f(x) =$ y

Functions as Strategies

I	$G^{\{\}}_{(f)}$	II
x_0		
	$\searrow$	
		y_0
	$\swarrow$	
x_1		
	$\searrow$	
		y_1
	$\swarrow$	
x_2		
	$\searrow$	

II wins iff $f(x) =$ y

Functions as Strategies

$$\begin{array}{ccc} \text{I} & G^{\{\}}_{(f)} & \text{II} \\ x_0 & & \\ & \searrow & \\ & & y_0 \\ & \swarrow & \\ x_1 & & \\ & \searrow & \\ & & y_1 \\ & \swarrow & \\ x_2 & & \\ & \searrow & \\ & & y_2 \end{array}$$

II wins iff $f(x) = $ y

Functions as Strategies

I	$G^{\{\}}_{(f)}$	II
x_0		
	$\searrow$	
		y_0
	$\swarrow$	
x_1		
	$\searrow$	
		y_1
	$\swarrow$	
x_2		
$\vdots$	$\searrow$	
		y_2
		$\vdots$

II wins iff $f(x) =$ y

Functions as Strategies

I	$G^{\{\}}_{(f)}$	II
x_0		
	$\searrow$	
		y_0
	$\swarrow$	
x_1		
	$\searrow$	
		y_1
	$\swarrow$	
x_2		
$\vdots$	$\searrow$	
		y_2
x_n		$\vdots$

II wins iff $f(x) =$ y

Functions as Strategies

$$\begin{array}{ccc} \text{I} & G^{\{\}}_{(f)} & \text{II} \\ x_0 & & \\ & \searrow & \\ & & y_0 \\ & \swarrow & \\ x_1 & & \\ & \searrow & \\ & & y_1 \\ & \swarrow & \\ x_2 & & \\ \vdots & \searrow & \\ & & y_2 \\ & & \vdots \\ x_n & & \\ & \searrow & \end{array}$$

II wins iff $f(x) =$ y

Functions as Strategies

$$
\begin{array}{ccc}
\color{red}{\mathrm{I}} & G^{\{\}}_{(f)} & \color{blue}{\mathrm{II}} \\
\color{red}{x_0} & & \\
& \searrow & \\
& & \color{blue}{y_0} \\
& \swarrow & \\
\color{red}{x_1} & & \\
& \searrow & \\
& & \color{blue}{y_1} \\
& \swarrow & \\
\color{red}{x_2} & & \\
\vdots & \searrow & \\
& & \color{blue}{y_2} \\
\vdots & & \vdots \\
\color{red}{x_n} & & \\
& \searrow & \vdots \\
& & \color{blue}{y_n}
\end{array}
$$

II wins iff $f(x) = \mathrm{y}$

Functions as Strategies

I	$G^{\{\}}_{(f)}$	II
x_0		
	$\searrow$	
		y_0
	$\swarrow$	
x_1		
	$\searrow$	
		y_1
	$\swarrow$	
x_2		
$\vdots$	$\searrow$	
		y_2
		$\vdots$
x_n		
	$\searrow$	
		y_n
	$\swarrow$	

II wins iff $f(x) = $ y

Functions as Strategies

$$
\begin{array}{ccc}
\textbf{I} & G^{\{\}}_{(f)} & \textbf{II} \\
x_0 & & \\
 & \searrow & \\
 & & y_0 \\
 & \swarrow & \\
x_1 & & \\
 & \searrow & \\
 & & y_1 \\
 & \swarrow & \\
x_2 & & \\
\vdots & \searrow & \\
 & & y_2 \\
 & & \vdots \\
x_n & & \\
 & \searrow & \\
 & & y_n \\
 & \swarrow & \\
x_{n+1} & &
\end{array}
$$

II wins iff $f(x) =$ y

Functions as Strategies

I	$G^{\{\}}_{(f)}$	II
x_0		
	$\searrow$	
		y_0
	$\swarrow$	
x_1		
	$\searrow$	
		y_1
	$\swarrow$	
x_2		
$\vdots$	$\searrow$	
		y_2
		$\vdots$
x_n		
	$\searrow$	
		y_n
	$\swarrow$	
x_{n+1}		
	$\searrow$	

II wins iff $f(x) = $ y

Functions as Strategies

$$
\begin{array}{ccc}
\mathsf{I} & G^{\{\}}_{(f)} & \mathsf{II} \\
x_0 & & \\
 & \searrow & \\
 & & y_0 \\
 & \swarrow & \\
x_1 & & \\
 & \searrow & \\
 & & y_1 \\
 & \swarrow & \\
x_2 & & \\
\vdots & \searrow & \\
 & & y_2 \\
x_n & & \vdots \\
 & \searrow & \\
 & & y_n \\
 & \swarrow & \\
x_{n+1} & & \\
 & \searrow & \\
 & & y_{n+1}
\end{array}
$$

II wins iff $f(x) =$ y

Functions as Strategies

$$\begin{array}{ccc} \text{I} & G^{\{\}}_{(f)} & \text{II} \\ x_0 & & \\ & \searrow & \\ & & y_0 \\ & \swarrow & \\ x_1 & & \\ & \searrow & \\ & & y_1 \\ & \swarrow & \\ x_2 & & \\ \vdots & \searrow & \\ & & y_2 \\ & & \vdots \\ x_n & & \\ & \searrow & \\ & & y_n \\ & \swarrow & \\ x_{n+1} & & \\ \vdots & \searrow & \\ & & y_{n+1} \\ & & \vdots \end{array}$$

II wins iff $f(x) =$ y

Functions as Strategies

I	$G^{\{\}}_{(f)}$	II
x_0		
	$\searrow$	
		y_0
	$\swarrow$	
x_1		
	$\searrow$	
		y_1
	$\swarrow$	
x_2		
$\vdots$	$\searrow$	
		y_2
		$\vdots$
x_n		
	$\searrow$	
		y_n
	$\swarrow$	
x_{n+1}		
$\vdots$	$\searrow$	
		y_{n+1}
		$\vdots$
x		y

II wins iff $f(x) = $ y

Functions as Strategies

- $\mathbf{\Delta}^0_1$ means *"I know for sure after k steps"*
- $A \in \mathbf{\Delta}^0_1$, k *steps* decide $x \in A$
- $B = f^{-1}A \in \mathbf{\Delta}^0_1$, n *steps* decide $y \in B$
- n might be much larger than k
- requires to wait long enough

Functions as Strategies

I $\quad G^{\{s\}}_{(f)} \quad$ II

II wins iff $f(x) =$ y

William W. Wadge (1960's)

Functions as Strategies

I $\quad G^{\{s\}}_{(f)} \quad$ II

II wins iff $f(x) =$ y

Functions as Strategies

I $\quad G^{\{s\}}_{(f)} \quad$ II

x_0

II wins iff $f(x) = \mathsf{y}$

Functions as Strategies

I $\quad G^{\{s\}}_{(f)} \quad$ II

x_0

$\searrow$

II wins iff $f(x) =$ y

Functions as Strategies

I $\quad G^{\{s\}}_{(f)} \quad$ II

x_0

$\searrow$

s

II wins iff $f(x) =$ y

Functions as Strategies

I $\quad G^{\{s\}}_{(f)} \quad$ II

x_0

$\searrow$

s

$\swarrow$

II wins iff $f(x) =$ y

Functions as Strategies

I $G^{\{s\}}_{(f)}$ II

x_0

↘

s

↙

x_1

II wins iff $f(x) =$ y

Functions as Strategies

$$\mathbf{I} \quad G_{(f)}^{\{s\}} \quad \mathbf{II}$$

$$\begin{array}{ccc} x_0 & & \\ & \searrow & \\ & & s \\ & \swarrow & \\ x_1 & & \\ & \searrow & \end{array}$$

II wins iff $f(x) = y$

Functions as Strategies

$\mathbf{I}$ $\quad G^{\{s\}}_{(f)}$ $\quad \mathbf{II}$

x_0

$\searrow$

$\quad\quad s$

$\swarrow$

x_1

$\searrow$

$\quad\quad s$

II wins iff $f(x) =$ y

Functions as Strategies

$$\text{I} \quad G^{\{s\}}_{(f)} \quad \text{II}$$

$$\begin{array}{ccc}
x_0 & & \\
& \searrow & \\
& & s \\
& \swarrow & \\
x_1 & & \\
& \searrow & \\
& & s \\
& \swarrow & \\
\end{array}$$

II wins iff $f(x) = \text{y}$

Functions as Strategies

$\mathbf{I}$ $\quad G^{\{s\}}_{(f)}$ $\quad \mathbf{II}$

x_0

$\searrow$

s

$\swarrow$

x_1

$\searrow$

s

$\swarrow$

x_2

II wins iff $f(x) =$ y

Functions as Strategies

I $\quad G^{\{s\}}_{(f)} \quad$ II

x_0

$\searrow$

$\qquad s$

$\swarrow$

x_1

$\searrow$

$\qquad s$

$\swarrow$

x_2

$\searrow$

II wins iff $f(x) =$ y

Functions as Strategies

$$\text{I} \quad G^{\{s\}}_{(f)} \quad \text{II}$$

$$\begin{array}{lcl} x_0 & & \\ & \searrow & \\ & & s \\ & \swarrow & \\ x_1 & & \\ & \searrow & \\ & & s \\ & \swarrow & \\ x_2 & & \\ & \searrow & \\ & & s \end{array}$$

II wins iff $f(x) = \mathsf{y}$

Functions as Strategies

$$\text{I} \quad G_{(f)}^{\{s\}} \quad \text{II}$$

$$
\begin{array}{ccc}
x_0 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_1 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_2 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow &
\end{array}
$$

II wins iff $f(x) =$ y

Functions as Strategies

I $\quad G_{(f)}^{\{s\}} \quad$ II

x_0

$\searrow$

$\quad\quad s$

$\swarrow$

x_1

$\searrow$

$\quad\quad s$

$\swarrow$

x_2

$\searrow$

$\quad\quad s$

$\swarrow$

x_3

II wins iff $f(x) =$ y

Functions as Strategies

I $\quad G_{(f)}^{\{s\}} \quad$ II

x_0

$\searrow$

$\qquad s$

$\swarrow$

x_1

$\searrow$

$\qquad s$

$\swarrow$

x_2

$\searrow$

$\qquad s$

$\swarrow$

x_3

$\searrow$

II wins iff $f(x) =$ y

Functions as Strategies

I	$G^{\{s\}}_{(f)}$	II
x_0		
	$\searrow$	
		s
	$\swarrow$	
x_1		
	$\searrow$	
		s
	$\swarrow$	
x_2		
	$\searrow$	
		s
	$\swarrow$	
x_3		
	$\searrow$	
		y_1

II wins iff $f(x) =$ y

Functions as Strategies

I	$G_{(f)}^{\{s\}}$	II
x_0		
	$\searrow$	
		s
	$\swarrow$	
x_1		
	$\searrow$	
		s
	$\swarrow$	
x_2		
	$\searrow$	
		s
	$\swarrow$	
x_3		
	$\searrow$	
		y_1
	$\swarrow$	

II wins iff $f(x) = \mathsf{y}$

Functions as Strategies

I $\quad G^{\{s\}}_{(f)} \quad$ II

x_0

$\searrow$

$\qquad s$

$\swarrow$

x_1

$\searrow$

$\qquad s$

$\swarrow$

x_2

$\searrow$

$\qquad s$

$\swarrow$

x_3

$\searrow$

$\qquad y_1$

$\swarrow$

x_4

II wins iff $f(x) =$ y

Functions as Strategies

I	$G^{\{s\}}_{(f)}$	II
x_0		
	$\searrow$	
		s
	$\swarrow$	
x_1		
	$\searrow$	
		s
	$\swarrow$	
x_2		
	$\searrow$	
		s
	$\swarrow$	
x_3		
	$\searrow$	
		y_1
	$\swarrow$	
x_4		
	$\searrow$	

II wins iff $f(x) = \mathsf{y}$

Functions as Strategies

I $\quad G_{(f)}^{\{s\}} \quad$ II

x_0

$\searrow$

s

$\swarrow$

x_1

$\searrow$

s

$\swarrow$

x_2

$\searrow$

s

$\swarrow$

x_3

$\searrow$

y_1

$\swarrow$

x_4

$\searrow$

y_2

II wins iff $f(x) =$ y

Functions as Strategies

$$
\begin{array}{ccc}
\textbf{I} & G_{(f)}^{\{s\}} & \textbf{II} \\
x_0 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_1 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_2 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_3 & & \\
 & \searrow & \\
 & & y_1 \\
 & \swarrow & \\
x_4 & & \\
 & \searrow & \\
 & & y_2 \\
 & \swarrow &
\end{array}
$$

II wins iff $f(x) =$ y

Functions as Strategies

$$\begin{array}{ccc}
\text{I} & G^{\{s\}}_{(f)} & \text{II} \\
x_0 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_1 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_2 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_3 & & \\
 & \searrow & \\
 & & y_1 \\
 & \swarrow & \\
x_4 & & \\
 & \searrow & \\
 & & y_2 \\
 & \swarrow & \\
\hline
x & & y
\end{array}$$

II wins iff $f(x) =$ y

Functions as Strategies

- II has a w.s. in $\mathbb{G}^{\{s\}}(f) \iff f$ is continuous.
- ($\Longrightarrow$) Let $u\mathcal{N}$ be any basic open set.

$$f^{-1}u{\cdot}\mathcal{N} = \bigcup_{v \in P_u^I} v{\cdot}\mathcal{N}$$

  where P_u^I is the set of all positions v s.t. I is in position v when II is in position u.

- ($\Longleftarrow$) At first move, II considers the partition $\{f^{-1}\langle n\rangle \cdot\mathcal{N} :\ n \in \mathbb{N}\}$ of the Baire space into $\mathbf{\Delta}_1^0$ sets. And waits until I has reached a long enough position v that belongs to some $f^{-1}\langle n\rangle \cdot\mathcal{N}$. Then II plays this value n_0, and considers $\{f^{-1}\langle n_0, n\rangle \cdot\mathcal{N} :\ n \in \mathbb{N}\}\ldots$

Functions as Strategies

- $$\mathit{II} \text{ has a w.s. in } \mathbb{G}^{\{s\}}(f) \iff f \text{ is continuous}$$

- $$\mathit{II} \text{ has a w.s. in } \mathbb{G}^{\{\}}(f) \iff f \text{ is } \ldots$$

$$\forall \varepsilon \exists \alpha \ |x - x_0| < \alpha \Longrightarrow |f(x) - f(x_0)| < \varepsilon$$

$$\forall \varepsilon |x - x_0| < \varepsilon \Longrightarrow |f(x) - f(x_0)| < \varepsilon$$

Functions as Strategies

- f is 1-Lipschitz

$$\forall \varepsilon |x - x_0| < \varepsilon \Longrightarrow |f(x) - f(x_0)| < \varepsilon$$

- f is k-Lipschitz

$$\forall \varepsilon |x - x_0| < \varepsilon / k \Longrightarrow |f(x) - f(x_0)| < \varepsilon$$

Functions as Strategies

Definition

$$\mathit{II} \text{ has a w.s. in } \mathbb{G}^{\{s_1,s_2,\ldots,s_n\}}(f) \iff f \text{ is } n\text{−Lipschitz}$$

Theorem

f is countinuous $\iff$ f is n-Lipschitz for some $n \in \mathbb{N}$.

Definition

$$\mathit{II} \text{ has a w.s. in } \mathbb{G}^{\{s_1,s_2,\ldots,s_n\}}(f) \iff f \text{ is } n\text{–Lipschitz}$$

Theorem

f is countinuous $\Longleftrightarrow$ f is n-Lipschitz for some $n \in \mathbb{N}$.

Counting with Ordinals

Counting below 4

1. $0, 1, 2, 3$
2. $3, 2, 1, 0$

Counting below $\omega + 2$

1. $0, 1, 2, 3, \ldots \ldots$
2. $\omega + 1, \omega, n, n - 1, n - 2, \ldots, 3, 2, 1, 0$

Definition

$\mathbb{G}^{\{s_\xi\colon \xi<\alpha\}}(f)$ as $\mathbb{G}^{\{s\}}(f)$ except that $I\!I$ must produce a sequence of **strictly decreasing** s_ξ while skipping

Functions as Strategies

I $\quad G_{(f)}^{\{s_\xi \ :\xi<\alpha\}} \quad$ II

II wins iff $f(x) =$ y

Functions as Strategies

I $G_{(f)}^{\{s_\xi\ :\xi<\alpha\}}$ II

x_0

II wins iff $f(x)=$ y

Functions as Strategies

I $G_{(f)}^{\{s_\xi\ :\xi<\alpha\}}$ II

x_0

$\searrow$

II wins iff $f(x) =$ y

Functions as Strategies

I $G_{(f)}^{\{s_\xi \ :\xi<\alpha\}}$ II

x_0

↘

s_1

II wins iff $f(x) =$ y

Functions as Strategies

I $\quad G_{(f)}^{\{s_\xi \, : \xi < \alpha\}}$ $\quad \mathsf{II}$

x_0

$\searrow$

s_1

$\swarrow$

II wins iff $f(x) =$ y

Functions as Strategies

I	$G_{(f)}^{\{s_\xi\ :\xi<\alpha\}}$	II
x_0		
	$\searrow$	
		s_1
	$\swarrow$	
x_1		

II wins iff $f(x) =$ y

Functions as Strategies

$\mathbf{I}$ $\quad G_{(f)}^{\{s_\xi \ :\xi<\alpha\}}$ $\quad \mathbf{II}$

x_0

$\searrow$

s_1

$\swarrow$

x_1

$\searrow$

II wins iff $f(x) =$ y

Functions as Strategies

$$\begin{array}{ccc} \mathbf{I} & G_{(f)}^{\{s_\xi\ :\xi<\alpha\}} & \mathbf{II} \\ x_0 & & \\ & \searrow & \\ & & s_1 \\ & \swarrow & \\ x_1 & & \\ & \searrow & \\ & & s_0 \end{array}$$

II wins iff $f(x) = $ y

Functions as Strategies

I	$G_{(f)}^{\{s_\xi\ :\xi<\alpha\}}$	II
x_0		
	$\searrow$	
		s_1
	$\swarrow$	
x_1		
	$\searrow$	
		s_0
	$\swarrow$	

II wins iff $f(x) =$ y

Functions as Strategies

I	$G_{(f)}^{\{s_\xi\ :\xi<\alpha\}}$	II
x_0		
	$\searrow$	
		s_1
	$\swarrow$	
x_1		
	$\searrow$	
		s_0
	$\swarrow$	
x_2		

II wins iff $f(x) =$ y

Functions as Strategies

I	$G_{(f)}^{\{s_\xi\ :\xi<\alpha\}}$	II
x_0		
	$\searrow$	
		s_1
	$\swarrow$	
x_1		
	$\searrow$	
		s_0
	$\swarrow$	
x_2		
	$\searrow$	

II wins iff $f(x) =$ y

Functions as Strategies

I	$G_{(f)}^{\{s_\xi\ :\xi<\alpha\}}$	II
x_0		
	$\searrow$	
		s_1
	$\swarrow$	
x_1		
	$\searrow$	
		s_0
	$\swarrow$	
x_2		
	$\searrow$	
		y_0

II wins iff $f(x) =$ y

Functions as Strategies

I	$G_{(f)}^{\{s_\xi\ :\xi<\alpha\}}$	II
x_0		
	$\searrow$	
		s_1
	$\swarrow$	
x_1		
	$\searrow$	
		s_0
	$\swarrow$	
x_2		
	$\searrow$	
		y_0
	$\swarrow$	

II wins iff $f(x) =$ y

Functions as Strategies

I	$G_{(f)}^{\{s_\xi\ :\xi<\alpha\}}$	II
x_0		
	$\searrow$	
		s_1
	$\swarrow$	
x_1		
	$\searrow$	
		s_0
	$\swarrow$	
x_2		
	$\searrow$	
		y_0
	$\swarrow$	
x_3		

II wins iff $f(x) =$ y

Functions as Strategies

I	$G_{(f)}^{\{s_\xi \ : \xi < \alpha\}}$	II
x_0		
	$\searrow$	
		s_1
	$\swarrow$	
x_1		
	$\searrow$	
		s_0
	$\swarrow$	
x_2		
	$\searrow$	
		y_0
	$\swarrow$	
x_3		
	$\searrow$	

II wins iff $f(x) =$ y

Functions as Strategies

I	$G_{(f)}^{\{s_\xi\ :\xi<\alpha\}}$	II
x_0		
	$\searrow$	
		s_1
	$\swarrow$	
x_1		
	$\searrow$	
		s_0
	$\swarrow$	
x_2		
	$\searrow$	
		y_0
	$\swarrow$	
x_3		
	$\searrow$	
		$s_{\omega+1}$

II wins iff $f(x) =$ y

Functions as Strategies

I	$G_{(f)}^{\{s_\xi\ :\xi<\alpha\}}$	II
x_0		
	$\searrow$	
		s_1
	$\swarrow$	
x_1		
	$\searrow$	
		s_0
	$\swarrow$	
x_2		
	$\searrow$	
		y_0
	$\swarrow$	
x_3		
	$\searrow$	
		$s_{\omega+1}$
	$\swarrow$	

II wins iff $f(x) =$ y

Functions as Strategies

I	$G_{(f)}^{\{s_\xi\ :\xi<\alpha\}}$	II
x_0		
	$\searrow$	
		s_1
	$\swarrow$	
x_1		
	$\searrow$	
		s_0
	$\swarrow$	
x_2		
	$\searrow$	
		y_0
	$\swarrow$	
x_3		
	$\searrow$	
		$s_{\omega+1}$
	$\swarrow$	
x_4		

II wins iff $f(x) = \mathrm{y}$

Functions as Strategies

I	$G_{(f)}^{\{s_\xi : \xi<\alpha\}}$	II
x_0		
	$\searrow$	
		s_1
	$\swarrow$	
x_1		
	$\searrow$	
		s_0
	$\swarrow$	
x_2		
	$\searrow$	
		y_0
	$\swarrow$	
x_3		
	$\searrow$	
		$s_{\omega+1}$
	$\swarrow$	
x_4		
	$\searrow$	

II wins iff $f(x) = $ y

Functions as Strategies

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x_0		
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		s_1
	$\swarrow$	
x_1		
	$\searrow$	
		s_0
	$\swarrow$	
x_2		
	$\searrow$	
		y_0
	$\swarrow$	
x_3		
	$\searrow$	
		$s_{\omega+1}$
	$\swarrow$	
x_4		
	$\searrow$	
		s_ω

II wins iff $f(x) =$ y

Functions as Strategies

I	$G_{(f)}^{\{s_\xi\ :\xi<\alpha\}}$	II
x_0		
	$\searrow$	
		s_1
	$\swarrow$	
x_1		
	$\searrow$	
		s_0
	$\swarrow$	
x_2		
	$\searrow$	
		y_0
	$\swarrow$	
x_3		
	$\searrow$	
		$s_{\omega+1}$
	$\swarrow$	
x_4		
	$\searrow$	
		s_ω
	$\swarrow$	

II wins iff $f(x) =$ y

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		s_0
	$\swarrow$	
x_2		
	$\searrow$	
		y_0
	$\swarrow$	
x_3		
	$\searrow$	
		$s_{\omega+1}$
	$\swarrow$	
x_4		
	$\searrow$	
		s_ω
	$\swarrow$	

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		s_1
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x_1		
	$\searrow$	
		s_0
	$\swarrow$	
x_2		
	$\searrow$	
		y_0
	$\swarrow$	
x_3		
	$\searrow$	
		$s_{\omega+1}$
	$\swarrow$	
x_4		
	$\searrow$	
		s_ω
	$\swarrow$	
x_5		

II wins iff $f(x) =$ y

Functions as Strategies

I	$G_{(f)}^{\{s_\xi\ :\xi<\alpha\}}$	II
x_0		
	$\searrow$	
		s_1
	$\swarrow$	
x_1		
	$\searrow$	
		s_0
	$\swarrow$	
x_2		
	$\searrow$	
		y_0
	$\swarrow$	
x_3		
	$\searrow$	
		$s_{\omega+1}$
	$\swarrow$	
x_4		
	$\searrow$	
		s_ω
	$\swarrow$	
x_5		
	$\searrow$	

II wins iff $f(x) =$ y

I	$G_{(f)}^{\{s_\xi\ :\xi<\alpha\}}$	II
x_0		
	$\searrow$	
		s_1
	$\swarrow$	
x_1		
	$\searrow$	
		s_0
	$\swarrow$	
x_2		
	$\searrow$	
		y_0
	$\swarrow$	
x_3		
	$\searrow$	
		$s_{\omega+1}$
	$\swarrow$	
x_4		
	$\searrow$	
		s_ω
	$\swarrow$	
x_5		
	$\searrow$	
		s_1

II wins iff $f(x) =$ y

Functions as Strategies

I	$G_{(f)}^{\{s_\xi\ :\xi<\alpha\}}$	II
x_0		
	$\searrow$	
		s_1
	$\swarrow$	
x_1		
	$\searrow$	
		s_0
	$\swarrow$	
x_2		
	$\searrow$	
		y_0
	$\swarrow$	
x_3		
	$\searrow$	
		$s_{\omega+1}$
	$\swarrow$	
x_4		
	$\searrow$	
		s_ω
	$\swarrow$	
x_5		
	$\searrow$	
		s_1
	$\swarrow$	

II wins iff $f(x) =$ y

Functions as Strategies

I	$G_{(f)}^{\{s_\xi \,:\xi<\alpha\}}$	II
x_0	$\searrow$	
	$\swarrow$	s_1
x_1	$\searrow$	
	$\swarrow$	s_0
x_2	$\searrow$	
	$\swarrow$	y_0
x_3	$\searrow$	
	$\swarrow$	$s_{\omega+1}$
x_4	$\searrow$	
	$\swarrow$	s_ω
x_5	$\searrow$	
	$\swarrow$	s_1
x_6		

II wins iff $f(x) =$ y

Functions as Strategies

I	$G_{(f)}^{\{s_\xi\ :\xi<\alpha\}}$	II
x_0		
	$\searrow$	
		s_1
	$\swarrow$	
x_1		
	$\searrow$	
		s_0
	$\swarrow$	
x_2		
	$\searrow$	
		y_0
	$\swarrow$	
x_3		
	$\searrow$	
		$s_{\omega+1}$
	$\swarrow$	
x_4		
	$\searrow$	
		s_ω
	$\swarrow$	
x_5		
	$\searrow$	
		s_1
	$\swarrow$	
x_6		
	$\searrow$	

II wins iff $f(x) =$ y

Functions as Strategies

I	$G_{(f)}^{\{s_\xi : \xi < \alpha\}}$	II
x_0		
	$\searrow$	
		s_1
	$\swarrow$	
x_1		
	$\searrow$	
		s_0
	$\swarrow$	
x_2		
	$\searrow$	
		y_0
	$\swarrow$	
x_3		
	$\searrow$	
		$s_{\omega+1}$
	$\swarrow$	
x_4		
	$\searrow$	
		s_ω
	$\swarrow$	
x_5		
	$\searrow$	
		s_1
	$\swarrow$	
x_6		
	$\searrow$	
		s_0

II wins iff $f(x) =$ y

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x_0		
	$\searrow$	
		s_1
	$\swarrow$	
x_1		
	$\searrow$	
		s_0
	$\swarrow$	
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	$\searrow$	
		s_ω
	$\swarrow$	
x_5		
	$\searrow$	
		s_1
	$\swarrow$	
x_6		
	$\searrow$	
		s_0
	$\swarrow$	

II wins iff $f(x) =$ y

Functions as Strategies

I	$G_{(f)}^{\{s_\xi\ :\xi<\alpha\}}$	II
x_0		
	$\searrow$	
		s_1
	$\swarrow$	
x_1		
	$\searrow$	
		s_0
	$\swarrow$	
x_2		
	$\searrow$	
		y_0
	$\swarrow$	
x_3		
	$\searrow$	
		$s_{\omega+1}$
	$\swarrow$	
x_4		
	$\searrow$	
		s_ω
	$\swarrow$	
x_5		
	$\searrow$	
		s_1
	$\swarrow$	
x_6		
	$\searrow$	
		s_0
	$\swarrow$	
x_7		

II wins iff $f(x) =$ y

Functions as Strategies

I	$G_{(f)}^{\{s_\xi\ :\xi<\alpha\}}$	II
x_0		
	$\searrow$	
		s_1
	$\swarrow$	
x_1		
	$\searrow$	
		s_0
	$\swarrow$	
x_2		
	$\searrow$	
		y_0
	$\swarrow$	
x_3		
	$\searrow$	
		$s_{\omega+1}$
	$\swarrow$	
x_4		
	$\searrow$	
		s_ω
	$\swarrow$	
x_5		
	$\searrow$	
		s_1
	$\swarrow$	
x_6		
	$\searrow$	
		s_0
	$\swarrow$	
x_7		
	$\searrow$	

II wins iff $f(x) =$ y

Functions as Strategies

I	$G_{(f)}^{\{s_\xi\ :\xi<\alpha\}}$	II
x_0		
	$\searrow$	
		s_1
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x_1		
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		s_0
	$\swarrow$	
x_2		
	$\searrow$	
		y_0
	$\swarrow$	
x_3		
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		$s_{\omega+1}$
	$\swarrow$	
x_4		
	$\searrow$	
		s_ω
	$\swarrow$	
x_5		
	$\searrow$	
		s_1
	$\swarrow$	
x_6		
	$\searrow$	
		s_0
	$\swarrow$	
x_7		
	$\searrow$	
		y_1

II wins iff $f(x) =$ y

Functions as Strategies

I	$G_{(f)}^{\{s_\xi\ :\xi<\alpha\}}$	II
x_0		
		s_1
x_1		
		s_0
x_2		
		y_0
x_3		
		$s_{\omega+1}$
x_4		
		s_ω
x_5		
		s_1
x_6		
		s_0
x_7		
		y_1
x		y

II wins iff $f(x) = \mathrm{y}$

α-Lipschitz

Definition

f is α-Lipschitz $\iff$ *II* has a w.s. in $\mathbb{G}^{\{s_\xi:\ \xi<\alpha\}}(f)$

Theorem

f is countinuous $\iff$ f is α-Lipschitz for some $\alpha < \omega_1$.

What is the use of depicting a continuous function as a strategy?

Reduction

Definition

- $X \;\leq\; Y \quad \Longleftrightarrow \quad \exists f$ *simple* $(x \in X \Leftrightarrow f(x) \in Y)$
- Y is $\mathcal{C}-$**complete** $\Longleftrightarrow \begin{cases} Y \in \mathcal{C} \\ X \;\leq\; Y \text{ , any } X \in \mathcal{C} \end{cases}$

- X is less complicated than Y

Reduction Games

Definition

- $X \leq_W Y \iff \exists f$ *simple* $(x \in X \Leftrightarrow f(x) \in Y)$
  $\iff$ $I\!I$ **has a w.s. in** $\mathbb{W}(\mathbf{X}, \mathbf{Y})$

Reduction

Definition

- $X \leq Y \iff \exists f$ *simple* $(x \in X \Leftrightarrow f(x) \in Y)$

- *simple* w.r. to topological complexity means **continuous**

Wadge Ordering

Definition

- $X \leq_W Y \iff \exists f$ *continuous* $(x \in X \Leftrightarrow f(x) \in Y)$
  $\iff$ *II* **has a w.s. in** $\mathbb{W}(\mathbf{X}, \mathbf{Y})$

Wadge Ordering

Definition

- $X \leq_w Y \iff \exists f$ *continuous* $(x \in X \Leftrightarrow f(x) \in Y)$
  $\iff$ $I\!I$ **has a w.s. in** $\mathbb{W}(\mathbf{X}, \mathbf{Y})$
- Y is $\mathcal{C}-$**complete** $\iff \begin{cases} Y \in \mathcal{C} \\ X \leq_w Y, \text{ any } X \in \mathcal{C} \end{cases}$
- $\mathcal{C}$ is a **Wadge Class** $\iff$ some $Y \in \mathcal{C}$ is $\mathcal{C}-$**complete**

- $X \leq_w Y \iff X^{\complement} \leq_w Y^{\complement}$

Basic Properties of the Wadge Ordering

Lemma

- $\leq_W$ *is reflexive* $X \leq_W X$
- $\leq_W$ *is transitive* $X \leq_W Y \leq_W Z \implies X \leq_W Z$

Basic Properties of the Wadge Ordering

Lemma

- $\leq_w$ *is reflexive* $X \leq_w X$
- $\leq_w$ *is transitive* $X \leq_w Y \leq_w Z \implies X \leq_w Z$

- $\leq_w$ is a partial ordering
- with Determinacy ......

Definition

- $X \equiv_w Y \iff X \leq_w Y \leq_w X$
- $X <_w Y \iff X \leq_w Y \not\leq_w X$

Wadge Determinacy

Theorem (Louveau, Saint-Raymond (86))

$$\textit{Borel Determinacy} \overset{\Rightarrow}{\not\Leftarrow} \textit{Borel Wadge Determinacy}$$

Theorem (Harrington (78))

$$\Sigma^1_1\textit{-Determinacy} \iff \Sigma^1_1\textit{-Wadge Determinacy}$$

Theorem (Martin, Monk, Wadge)

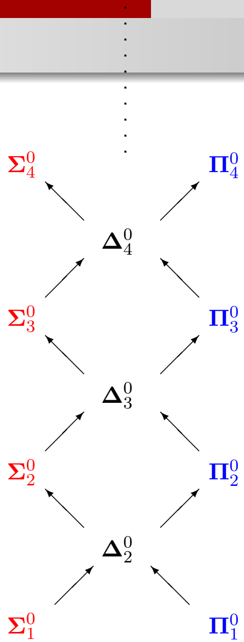
Assuming Determinacy,

- $<_W$ *is wellfounded, i.e. there is no*

$$X_0 {}_W\!> X_1 {}_W\!> X_2 {}_W\!> \dots {}_W\!> X_k {}_W\!> X_{k+1} {}_W\!> \dots$$

- *there are* **no three** *incomparable* X, Y, Z
- $X \not\leq_W Y \implies Y \leq_W X^{\complement}$

Wadge Ordering



Wadge Ordering

- $\leq_W$ yields a hierarchy on Borel sets: the Wadge hierarchy, a huge refinement of the Borel hierarchy
- $\mathbf{\Sigma}^0_2,\ \mathbf{\Pi}^0_2, \quad \rightsquigarrow \quad \omega_1$
- $\mathbf{\Sigma}^0_3,\ \mathbf{\Pi}^0_3\ , \quad \rightsquigarrow \quad \omega_1^{\omega_1}$
- $\mathbf{\Sigma}^0_4,\ \mathbf{\Pi}^0_4\ , \quad \rightsquigarrow \quad \omega_1^{\omega_1^{\omega_1}}$
- $\mathbf{\Sigma}^0_\omega,\ \mathbf{\Pi}^0_\omega\ , \quad \rightsquigarrow \quad \gg\ \sup_{n\in\mathbb{N}}\ \underbrace{\omega_1^{\omega_1^{\omega_1^{\cdot^{\cdot^{\cdot^{\omega_1}}}}}}}_{n}$

From Wadge Games to Conciliatory Games

Conciliatory sets are of the form $B \subseteq \mathbb{N}^{\leq\omega}$

- From $B \subseteq \mathbb{N}^{\leq\omega}$, one defines the set $B^b \subseteq (\mathbb{N} \cup \{b\})^\omega$ of infinite sequences by:

$$B^b = \left\{ x \in (\mathbb{N} \cup \{b\})^\omega \mid x_{[\ /b]} \in B \right\},$$

- We use no topology on conciliatory sets, but for any pointclass Γ, we say

$$B \subseteq \mathbb{N}^{\leq\omega} \text{ is in } \Gamma \iff B^b \in \Gamma.$$

.

Conciliatory Game

Given $A, B \subseteq \mathbb{N}^{\leq\omega}$. The conciliatory form of the Wadge Game:

$$\mathbf{C}(\mathbf{A}, \mathbf{B})$$

- same as in the Wadge game (I begins, II answers, etc.)
- except I can also skip, and both players do not have to produce an infinite sequence, so that at the end of the game, they have played respectively
$$x \in \mathbb{N}^{\leq\omega} \quad \text{and} \quad y \in \mathbb{N}^{\leq\omega}$$
- II wins if and only if $(x \in A \Leftrightarrow y \in B)$
- $A \leq_c B$ if and only if II has a winning strategy in $C(A, B)$

.

Conciliatory Game

Given $A, B \subseteq \mathbb{N}^{\leq\omega}$.

- II has a w.s. in $C(A, B) \Leftrightarrow \mathrm{II}$ has a w.s. in $W(A^b, B^b)$
- I has a w.s. in $C(A, A^{\complement}) \quad \Longrightarrow$ no conciliatory selfdual set

.

Conciliatory Game

Theorem

With enough determinacy

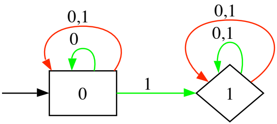
$$A \subseteq \mathbb{N}^\omega \text{ is non-selfdual}$$
$$\Longleftrightarrow$$
$$\text{there exists } B \subseteq \mathbb{N}^{\leq\omega} \text{ such that } A \equiv_W B^b$$
$$\Longleftrightarrow$$
$$\text{there exists } A_{finite} \subseteq \mathbb{N}^{<\omega} \text{ such that } A \equiv_W (A \cup A_{finite})^b$$

- The Wadge hierarchy restricted to *non-selfdual* sets is isomorphic to the conciliatory hierarchy, via ^{b}.
- The conciliatory hierarchy and the one induced by reductions by relatively continuous relations on the space $\mathbb{N}^{\leq\omega}$ endowed with the prefix topology are the same (Yann Pequignot).
- **Open Question**. Is there a topology on $\mathbb{N}^{\leq\omega}$ such that $\leq_c$ coincides with the reduction by continuous functions?

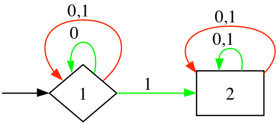
Wadge Ordering

In Automata Theory...

$\mathbf{\Pi}_1^0$–*complete*

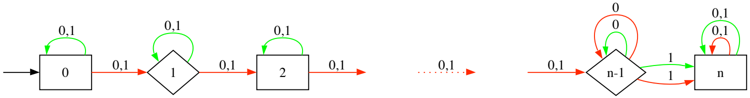


$\mathbf{\Sigma}_1^0$–*complete*

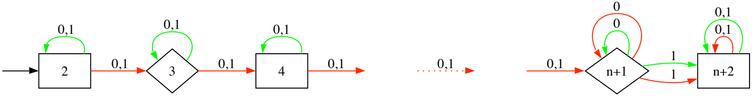


Wadge Ordering

$\mathbf{\Pi}^0_n$ $-complete$

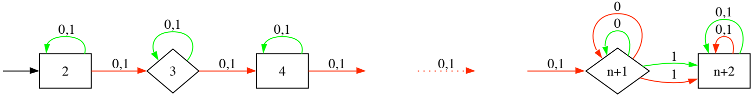


$\mathbf{\Pi}^0_n$ $-complete$



Wadge Ordering

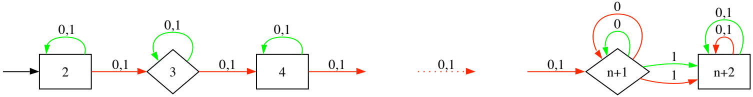
$\Pi_n^0 - complete$



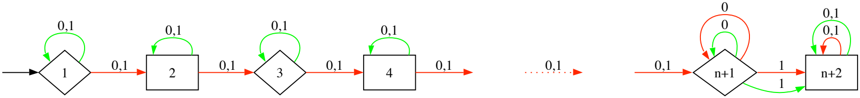
Given any $A \in \Pi_n^0$,

$$A \leq_W \Pi_n^0 - complete$$

Π^0_n *−complete*



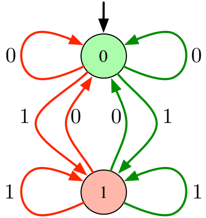
Σ^0_{n+1} *−complete*

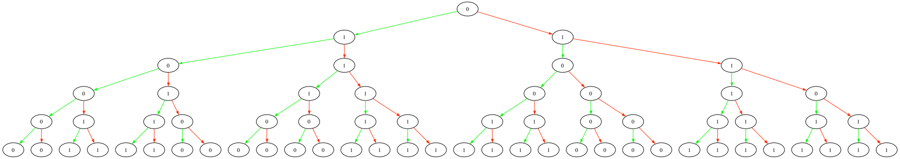


Unambiguous Tree Languages

Unambiguous Tree Languages

- Automata that read infinite (binary) trees





Unambiguous Tree Languages

- *nondeterministic parity tree automaton*

$$\mathcal{A} = \langle A, Q, I, \delta, r \rangle$$

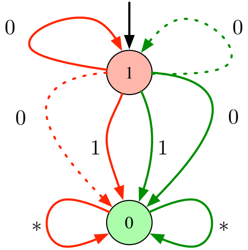
  - a finite input alphabet A,
  - a finite set Q of states,
  - a set of initial states $I \subseteq Q$,
  - a transition relation $\delta \subseteq Q \times A \times Q \times Q$ and
  - a priority function $r : Q \to \omega$.

- *Unambiguous automaton*

  nondeterministic with at most one accepting run on each input

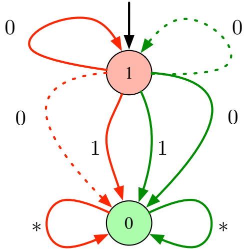
Unambiguous Tree Languages

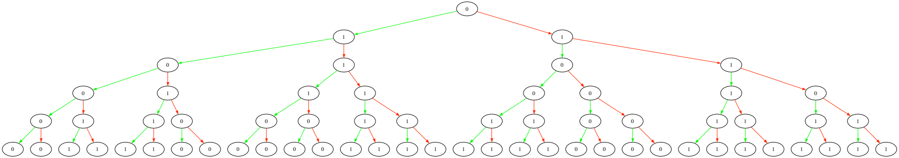
- Acceptance or Rejection of an input relies on *solving a game*
- two players:
  - *Automaton* takes charge of *nondeterminism*
  - *Pathfinder* takes charge of *branching*
- yields $\mathbf{\Delta}^1_2$ languages



Unambiguous Tree Languages

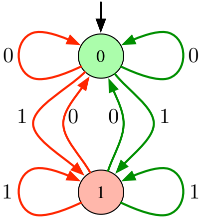
- What is the language?

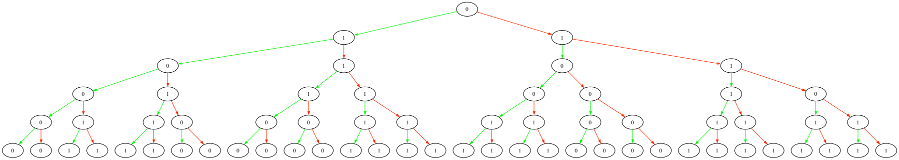




Unambiguous Tree Languages

- What is the language?





Unambiguous Tree Languages

- The space T_A (of all full binary trees on the alphabet A) equipped with the standard Cantor topology is a Polish space and is in fact homeomorphic to the Cantor space
- Let $L, M \subseteq T_A$,
  - in the Wadge game $W(L, M)$ each player builds a tree, say t_I and t_{II}.
  - At every round, player I plays first, and both players add a finite number of children to the terminal nodes of their tree.
  - Player II is allowed to skip its turn, but has to produce a tree in T_A throughout a game.
  - Player II wins the game if and only if $t_I \in L \Leftrightarrow t_{II} \in M$.
- $L \leq_W M \Leftrightarrow II$ has a winning strategy in the game $W(L, M)$

- A *conciliatory* binary tree over some alphabet A is a partial function

$$t : \{0,1\}^* \to A$$

with a prefix closed domain. Those trees can have both infinite and finite branches.

- For conciliatory languages L, M we define the *conciliatory* version of the Wadge game: $C(L, M)$
  - rules are similar, except that both player are allowed to skip and to produce trees with finite branches - or even finite trees.
  - $L \leq_c M$ if and only if II has a winning strategy in the game $C(L, M)$.
  - $L \equiv_c M$ if and only if $L \leq_c M$ and $M \leq_c L$
  - The conciliatory hierarchy is thus the partial order induced by $\leq_c$ on the equivalence classes given by $\equiv_c$.

Unambiguous Tree Languages

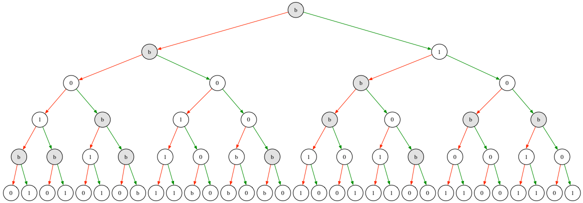
- From a conciliatory language L over A, one defines the corresponding language L^b of full trees over $A \cup \{b\}$ by:

$$L^b = \left\{ t \in T_{A \cup \{b\}} : t_{[\ /b]} \in L \right\},$$

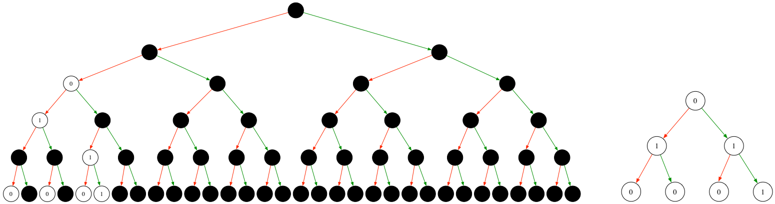
where b is an extra symbol that stands for "blank", and $t_{[\ /b]}$, the *undressing* of t, is informally the conciliatory tree over A obtained once all the occurences of b have been removed in a top-down manner.

Unambiguous Tree Languages

- A tree t with blanks



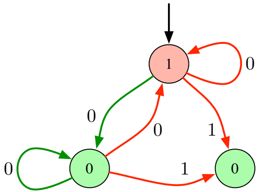
- The blanks are deleted in a top-down manner

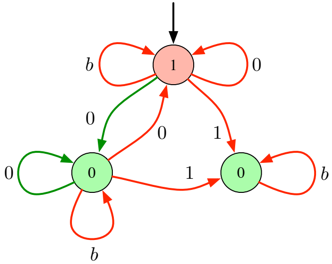


Lemma

Let L and M be conciliatory languages. Then

$$L \leq_c M \text{ if and only if } L^b \leq_W M^b.$$





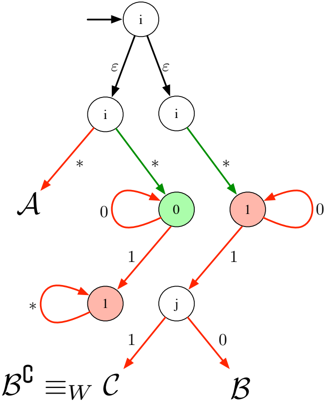
Operations on languages and their automatic counterparts

Unambiguous Tree Languages

- The sum ($+$)

Assuming enough determinacy:

$$d_c(\mathcal{B}+\mathcal{A}) = d_c(\mathcal{B}) + d_c(\mathcal{A}).$$



- The multiplication ($\bullet$)

$$\mathcal{A} \bullet n = \underbrace{\mathcal{A} + \mathcal{A} + \cdots + \mathcal{A}}_{n}$$

Assuming enough determinacy:

$$d_c(\mathcal{A} \bullet n) = d_c(\mathcal{A}) \cdot n.$$

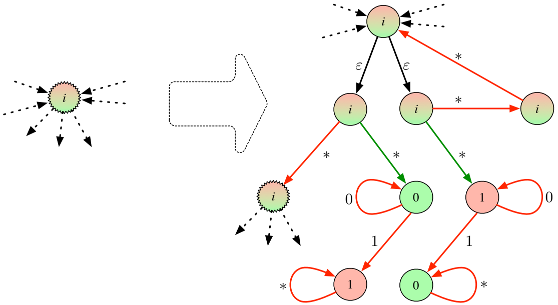
Lemma

If $\mathcal{A} <_c \mathcal{A}'$ and $\mathcal{B} \leq_c \mathcal{B}'$. Then, the following holds.

1. $\mathcal{A} + \mathcal{A} <_c \mathcal{B}' + \mathcal{A}'$
2. $\mathcal{A} <_c \mathcal{A} + \mathcal{B}$.

   In particular $\mathcal{A} \bullet n <_c \mathcal{A} \bullet (n+1)$ *for any* $0 < n < \omega$.

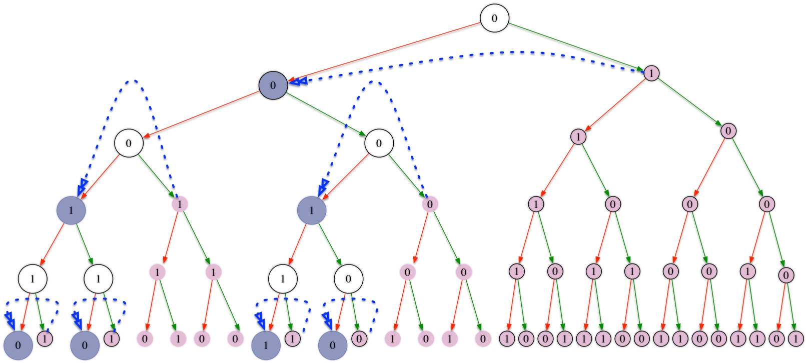
- The pseudo-exponentiation ($\boldsymbol{\omega}^{\mathcal{A}}$)



Assuming enough determinacy:

$$d_c(\boldsymbol{\omega}^{\mathcal{A}}) = \omega_1^{d_c(\mathcal{A})} + (1{,}0{,}{-1})$$

- The pseudo-exponentiation ($\omega^{\mathcal{A}}$)



Proposition

1. $\omega^{\left(\mathcal{A}^{\complement}\right)} \equiv_c \left(\omega^{\mathcal{A}}\right)^{\complement}$.
2. *If* $\mathcal{A} \leq_c \mathcal{B}$, *then* $\omega^{\mathcal{A}} \leq_c \omega^{\mathcal{B}}$.
3. *If* $\mathcal{A} <_c \mathcal{B}$, *then* $\omega^{\mathcal{A}} <_c \omega^{\mathcal{B}}$.
4. *Assuming enough determinacy,*

$$d_c(\omega^{\mathcal{A}}) = \omega_1^{d_c(\mathcal{A})+\varepsilon},$$

where

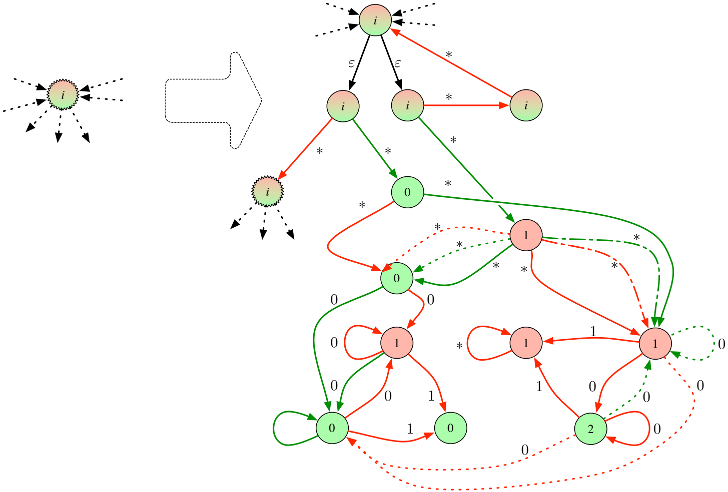
$$\varepsilon = \begin{cases} -1 & \text{if } d_c(\mathcal{A}) < \omega; \\ 0 & \text{if } d_c(\mathcal{A}) = \beta + n \text{ and } cof(\beta) = \omega_1; \\ 1 & \text{if } d_c(\mathcal{A}) = \beta + n \text{ and } cof(\beta) = \omega. \end{cases}$$

Proposition

If $\mathcal{A}, \mathcal{B} <_c \omega^{\mathcal{C}}$, *then*

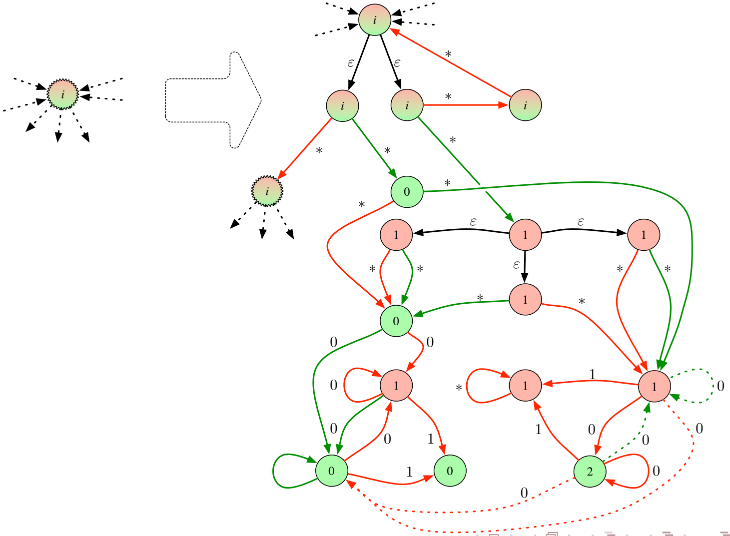
- $$\mathcal{A} + \mathcal{B} <_c \omega^{\mathcal{C}}$$
- *for any integer* n
$$\mathcal{A} \bullet n <_c \omega^{\mathcal{C}}$$

- The pseudo-epsilon ($\varepsilon_{\mathcal{A}}$)



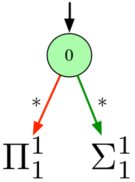
Unambiguous Tree Languages

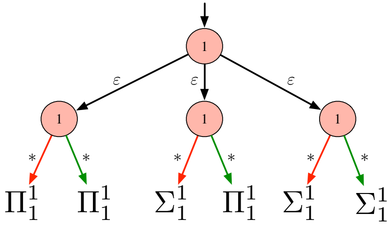
- The pseudo-epsilon ($\varepsilon_{\mathcal{A}}$)



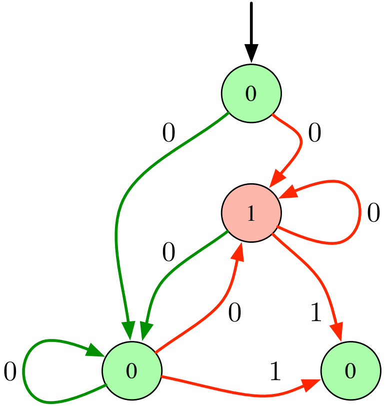
Unambiguous Tree Languages

- Auxiliary conditions for $\varepsilon_{\mathcal{A}}$





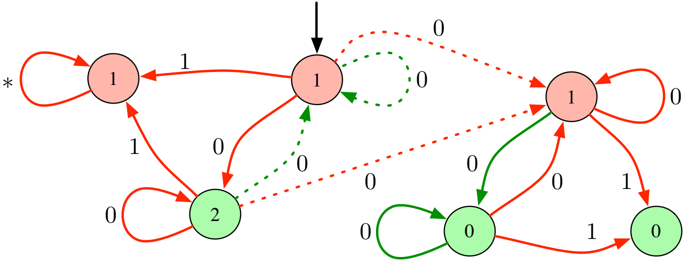
- An unambiguous (deterministic) $\mathbf{\Pi}^1_1$-complete



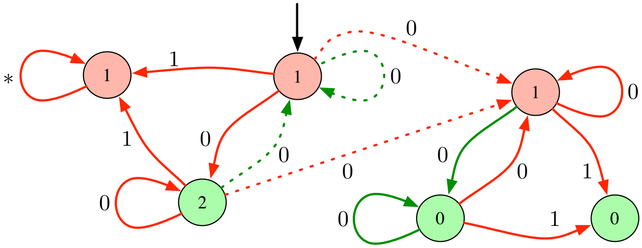
$\mathcal{A}_{\mathbf{\Pi}^1_1}$

Lemma (Szczepan Hummel)

there exists some unambiguous $\mathbf{\Sigma}^1_1$-complete tree automata



- *there exists a branch with 0's only that turns left infinitely many times*
- *look for the left most one!*



$\mathcal{A}_{\Sigma_1^1}$

Unambiguous Tree Languages

- *there exists a branch with 0's only that turns left infinitely many times*
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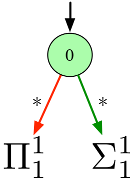
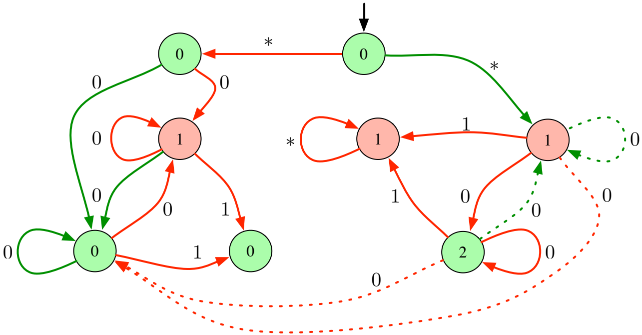
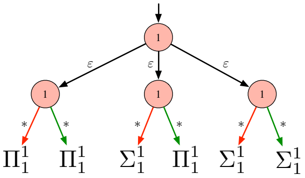


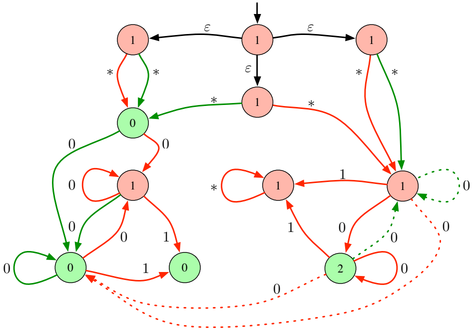


$\mathcal{A}_{D_2(\boldsymbol{\Pi}^1_1)}$

Unambiguous Tree Languages

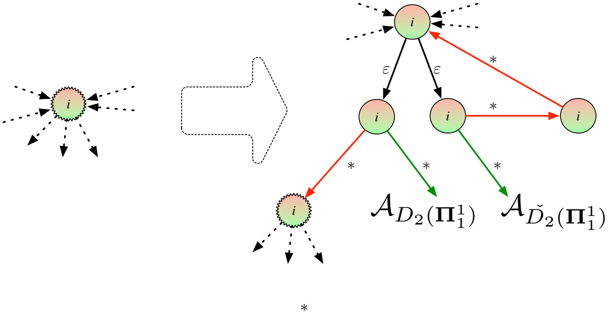
- *there exists a branch with 0's only that turns left infinitely many times*
- *look for the left most one!*





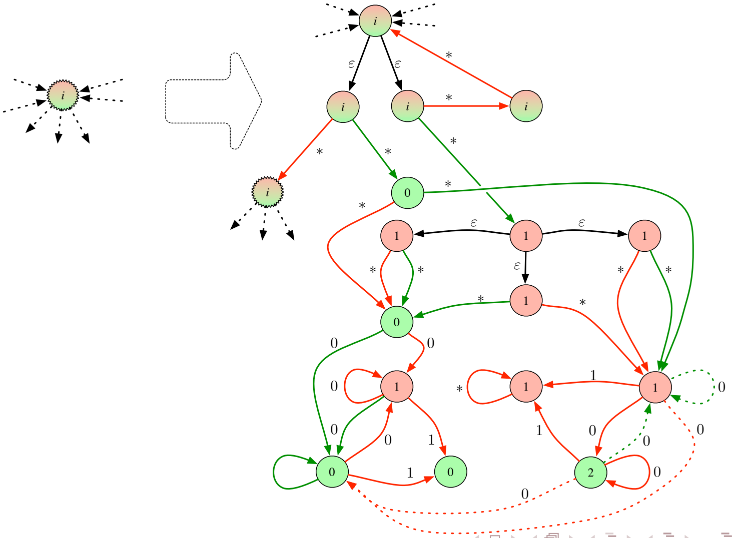
$\mathcal{A}_{\check{D}_2(\mathbf{\Pi^1_1})}$

- The pseudo-epsilon ($\varepsilon_{\mathcal{A}}$)



Unambiguous Tree Languages

- The pseudo-epsilon ($\varepsilon_{\mathcal{A}}$)



The pseudo-epsilon ($\varepsilon_{\mathcal{A}}$)

Theorem

If $\mathcal{A} \leq_c \mathcal{B}$, *then*

1. $$(\varepsilon_{\mathcal{A}})^{\complement} \equiv_c \varepsilon_{(\mathcal{A}^{\complement})}$$
2. $$\varepsilon_{\mathcal{A}} \leq_c \varepsilon_{\mathcal{B}}$$

If $\mathcal{A} <_c \mathcal{B}$, *then*

3. $$\varepsilon_{\mathcal{A}} <_c \varepsilon_{\mathcal{B}}$$
4. $$\omega^{(\varepsilon_{\mathcal{A}})} \equiv_c \varepsilon_{\mathcal{A}}$$

A fragment of the unambiguous Wadge hierarchy

Let us denote by Ω the first fixed point of the ordinal epsilon function, namely the one that enumerates the fixed points of the exponentiation of base ω:

- $\varepsilon_0 = \sup_{n<\omega} \underbrace{\omega^{\cdot^{\cdot^{\cdot^{\omega^0}}}}}_{n}$;
- $\varepsilon_{\alpha+1} = \sup_{n<\omega} \underbrace{\omega^{\cdot^{\cdot^{\cdot^{\omega^{(\varepsilon_\alpha+1)}}}}}}_{n}$;
- $\varepsilon_\lambda = \sup_{\alpha<\lambda} \varepsilon_\alpha$, for λ some limit ordinal.

Finally:

$$\Omega = \sup_{n<\omega} \underbrace{\varepsilon_{\cdot_{\cdot_{\cdot_{\varepsilon_0}}}}}_{n} .$$

A fragment of the unambiguous Wadge hierarchy

Another way to characterise Ω is to remember that an ordinal β is the set of its predecessors and notice that a non zero ordinal is of the form respectively

- $\beta = \omega^\alpha \Leftrightarrow \beta$ is closed under addition and
- $\beta = \varepsilon_\alpha \Leftrightarrow \beta$ is closed under $x \longmapsto \omega^x$.
- Ω is the first non null ordinal closed under
  - $x, y \longmapsto x + y$.
  - $x \longmapsto \omega^x$
  - $x \longmapsto \varepsilon_x$

Finally:

$$\Omega = \sup_{n<\omega} \underbrace{\varepsilon_{\cdot_{\cdot_{\cdot_{\varepsilon_0}}}}}_{n}.$$

A fragment of the unambiguous Wadge hierarchy

Every ordinal $\alpha > 0$ admits a unique Cantor Normal Form of base ω (CNF) which is an expression of the form

$$\alpha = \omega^{\alpha_k} \cdot n_k + \cdots + \omega^{\alpha_0} \cdot n_0$$

where

- $k < \omega$
- $0 < n_i < \omega$ (any $i \leq k$)
- $0 \leq \alpha_0 < \ldots < \alpha_k < \alpha$.

A fragment of the unambiguous Wadge hierarchy

For every ordinal $0 < \alpha < \Omega$, we inductively define a pair of unambiguous automata $(\mathcal{A}_\alpha, \bar{\mathcal{A}}_\alpha)$ whose languages are both non-selfdual and incomparable through the conciliatory ordering. If the CNF of α is $\alpha = \omega^{\alpha_k} \cdot n_k + \cdots + \omega^{\alpha_0} \cdot n_0$ we set

$$\mathcal{A}_\alpha = \mathcal{A}_{\omega^{\alpha_k}} \bullet n_k + \cdots + \mathcal{A}_{\omega^{\alpha_0}} \bullet n_0$$

and

$$\bar{\mathcal{A}}_\alpha = \mathcal{A}_{\omega^{\alpha_k}} \bullet n_k + \cdots + \bar{\mathcal{A}}_{\omega^{\alpha_0}} \bullet n_0,$$

where $\mathcal{A}_{\omega^{\alpha_i}}$ and $\bar{\mathcal{A}}_{\omega^{\alpha_i}}$ are respectively:

- $\ominus$ and $\oplus$ if $\alpha_i = 0$
- $\boldsymbol{\omega}^{\mathcal{A}_{\alpha_i}}$ and $\boldsymbol{\omega}^{\bar{\mathcal{A}}_{\alpha_i}}$ if $\alpha_i < \omega^{\alpha_i}$
- $\boldsymbol{\varepsilon}_{\mathcal{A}_{2+\beta}}$ and $\boldsymbol{\varepsilon}_{\bar{\mathcal{A}}_{2+\beta}}$ if $\alpha_i = \omega^{\alpha_i}$ and $\alpha_i = \varepsilon_\beta$ for some $\beta < \alpha_i$.

A fragment of the unambiguous Wadge hierarchy

Lemma

Let $0 < \alpha < \beta < \Omega$,

1. $\mathcal{A}_\alpha \not\leq_c \bar{\mathcal{A}}_\alpha$ *and* $\bar{\mathcal{A}}_\alpha \not\leq_c \mathcal{A}_\alpha$.
2.
   1. $\mathcal{A}_\alpha <_c \mathcal{A}_\beta$
   2. $\bar{\mathcal{A}}_\alpha <_c \mathcal{A}_\beta$
   3. $\mathcal{A}_\alpha <_c \bar{\mathcal{A}}_\beta$
   4. $\bar{\mathcal{A}}_\alpha <_c \bar{\mathcal{A}}_\beta$.

A fragment of the unambiguous Wadge hierarchy

Theorem

There exists a family $(\mathcal{A}_\alpha{}^b)_{\alpha<\Omega}$ *of unambiguous parity tree automata such that*

1. *they recognize languages of full trees over the alphabet* $\{0,1,b\}$*;*
2. $\alpha < \beta \Leftrightarrow \mathcal{A}_\alpha{}^b <_W \mathcal{A}_\beta{}^b$.
3. *their priorities are restricted to* $\{0,1,2\}$

Wadge hierarchy of all deterministic tree languages [Filip Murlak]:

$$(\omega^\omega)^3 + 3$$

$$\Omega = \sup_{n<\omega} \underbrace{\varepsilon^{\cdot^{\cdot^{\cdot^{\varepsilon_0}}}}}_{n}$$