The Fine Hierarchy of ω-Regular k-Partitions

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In [W79] K. Wagner gave in a sense the finest possible topological classification of regular ω-languages (i.e., of the subsets of X^ω for a finite alphabet X recognized by finite automata) known as the Wagner hierarchy. In particular, he completely described the (quotient structure of the) preorder $(\mathcal{R}; \leq_{CA})$ formed by the class $\mathcal{R}$ of regular subsets of X^ω and the reducibility by functions continuous in the Cantor topology on X^ω (note that in descriptive set theory the CA-reducibility is widely known as the Wadge reducibility).

In [S94, S95, S98] the Wagner hierarchy of regular ω-languages was related to the Wadge hierarchy and to the author's fine hierarchy [S95a]. This provided new proofs of results in [W79] and yielded some new results on the Wagner hierarchy. See also alternative algebraic approaches [CP97, CP99, DR06] and [CD09]. The aim of this paper is to generalize this theory from the case of regular ω-regular languages to the case of regular k-partitions of X^ω, i.e. k-tuples $(A_0, \ldots, A_{k-1})$ of pairwise disjoint regular sets satisfying $A_0 \cup \cdots \cup A_{k-1} = X^\omega$. Note that the ω-languages are in a bijective correspondence with 2-partitions of X^ω.

1) The structure $(\mathcal{R}; \leq_{CA})$ is almost well-ordered with the order type ω^ω, i.e. there are $A_\alpha \in \mathcal{R}$, $\alpha < \omega^\omega$, such that $A_\alpha <_{CA} A_\alpha \oplus \overline{A}_\alpha <_{CA} A_\beta$ for $\alpha < \beta < \omega^\omega$ and any regular set is CA-equivalent to one of the sets $A_\alpha, \overline{A}_\alpha, A_\alpha \oplus \overline{A}_\alpha (\alpha < \omega^\omega)$.
2) The CA-reducibility coincides on $\mathcal{R}$ with the DA-reducibility, i.e. the reducibility by functions computed by deterministic asynchronous finite transducers, and $\mathcal{R}$ is closed under the DA-reducibility.
3) Any level $\mathcal{R}_\alpha = \{C \mid C \leq_{DA} A_\alpha\}$ of the Wagner hierarchy is decidable.

A Muller k-acceptor is a pair $(\mathcal{A}, c)$ where $\mathcal{A}$ is an automaton and $c : C_{\mathcal{A}} \to k$ is a k-partition of $C_{\mathcal{A}} = \{f_{\mathcal{A}}(\xi) \mid \xi \in X^{\omega}\}$ where $f_{\mathcal{A}}(\xi)$ is the set of states which occur infinitely often in the sequence $f(i, \xi) \in Q^{\omega}$. Note that in this paper we consider only deterministic finite automata. Such a k-acceptor recognizes the k-partition $L(\mathcal{A}, c) = c \circ f_{\mathcal{A}}$ where $f_{\mathcal{A}} : X^{\omega} \to C_{\mathcal{A}}$ is the map defined above. We have the following characterization of the ω-regular partitions.

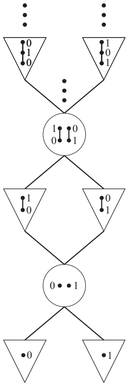
Proposition

A partition $L : X^{\omega} \to k$ is regular iff it is recognized by a Muller k-acceptor.

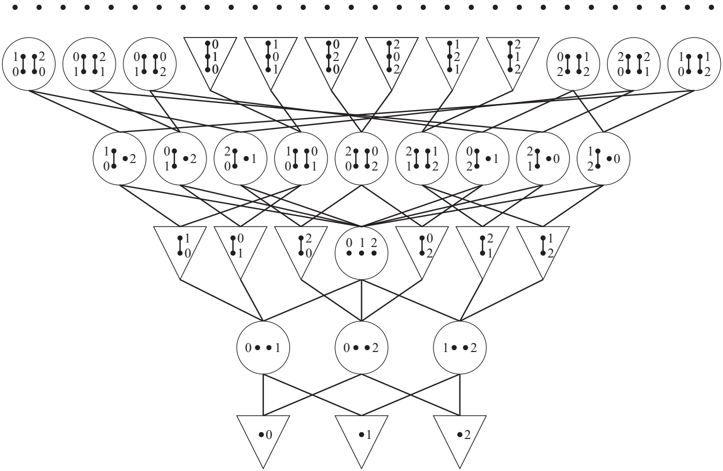
Let $(Q;\leq)$ be a poset. A *Q-poset* is a triple $(P,\leq,c)$ consisting of a finite nonempty poset $(P;\leq)$, $P\subseteq\omega$, and a labeling $c:P\to Q$. A *morphism* $f:(P,\leq,c)\to(P',\leq',c')$ of Q-posets is a monotone function $f:(P;\leq)\to(P';\leq')$ satisfying $\forall x\in P(c(x)\leq c'(f(x)))$.
Let $\mathcal{P}_Q$, $\mathcal{F}_Q$ and $\mathcal{T}_Q$ denote the sets of all finite Q-posets, Q-forests and Q-trees, respectively.
The *h-preorder* $\leq_h$ on $\mathcal{P}_Q$ is defined as follows: $P\leq_h P'$, if there is a morphism from P to P'. Note that for the particular case $Q=\bar{k}$ of the antichain with k elements we obtain the preorders $\mathcal{P}_k$, $\mathcal{F}_k$ and $\mathcal{T}_k$.

It is well known that if Q is a wqo then $(\mathcal{F}_Q;\leq_h)$ and $(\mathcal{T}_Q;\leq_h)$ are wqo's. Obviously, $P \subseteq Q$ implies $\mathbb{F}_P \subseteq \mathbb{F}_Q$, and $P \sqsubseteq Q$ (i.e., P is an initial segment of Q) implies $\mathbb{F}_P \sqsubseteq \mathbb{F}_Q$.
Define the sequence $\{\mathcal{F}_k(n)\}_{n<\omega}$ of preorders by induction on n as follows: $\mathcal{F}_k(0) = \overline{k}$ and $\mathcal{F}_k(n+1) = \mathcal{F}_{\mathcal{F}_k(n)}$. Identifying the elements $i < k$ of $\overline{k}$ with the corresponding minimal elements $s(i)$ of $\mathcal{F}_k(1)$, we may think that $\mathcal{F}_k(0) \sqsubseteq \mathcal{F}_k(1)$, hence $\mathcal{F}_k(n) \sqsubseteq \mathcal{F}_k(n+1)$ for each $n < \omega$ and $\mathcal{F}_k(\omega) = \bigcup_{n<\omega} \mathcal{F}_k(n)$ is a wqo.

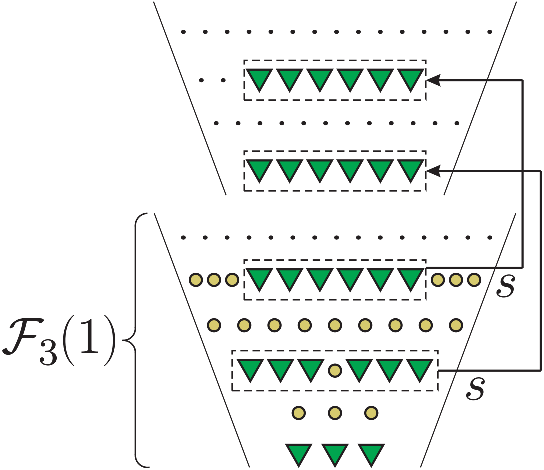
The preorders $\mathcal{F}_k(\omega)$, $\mathcal{T}_k(\omega)$ and the set $\mathcal{T}_k^{\sqcup}(\omega)$ of finite joins of elements in $\mathcal{T}_k(\omega)$, play an important role in the study of the FH of k-partitions because they provide convenient naming systems for the levels of this hierarchy (similar to the previous work where $\mathcal{F}_k$ and $\mathcal{T}_k$ where used to name the levels of the DH of k-partitions). Note that $\mathcal{F}_k(1) = \mathcal{F}_k$ and $\mathcal{T}_k(1) = \mathcal{T}_k$.
For the FH of ω-regular k-partitions, the structure $\mathcal{T}_k^{\sqcup}(2) = \mathcal{T}_{\mathcal{T}_k}^{\sqcup}$ is especially relevant. For $k = 2$ it is isomorphic to the structure of levels of the Wagner hierarchy.



Picture 1: An initial segment of $\mathcal{F}_2$.



Picture 2: An initial segment of $\mathcal{F}_3$.



Picture 3: A fragment of $\mathcal{T}_3^{\sqcup}(2)$.

Theorem

1. The quotient-posets of $(\mathcal{R}_k; \leq_{CA})$ and of $(\mathcal{R}_k; \leq_{DA})$ are isomorphic to the quotient-poset of $\mathcal{T}_k^{\sqcup}(2)$.
2. The relations $\leq_{CA}, \leq_{DA}$ coincide on $\mathcal{R}_k$, the same holds for the relations $\leq_{CS}, \leq_{DS}$.
3. The relations $L(\mathcal{A}, c) \leq_{CA} L(\mathcal{B}, d)$ and $L(\mathcal{A}, c) \leq_{DA} L(\mathcal{B}, d)$ are decidable.

1) Extending and modifying some operations of W. Wadge and A. Andretta on subsets of the Cantor space, we embed $\mathcal{T}_k^{\sqcup}(2)$ into $(\mathcal{R}_k;\leq_{CA})$ and $(\mathcal{R}_k;\leq_{DA})$ (an embedding is induced by $F\mapsto r(F)$).
2) We extend the author FH of sets [S98] to the FH of k-partitions over $(\mathbf{\Sigma}^0_1\cap\mathcal{R},\mathbf{\Sigma}^0_2\cap\mathcal{R})$ in such a way that $r(F)$ is CA-complete in $\mathbf{\Sigma}(F)$ and DA-complete in $\mathbf{\Sigma}\mathcal{R}(F)$.
3) Relate to any Muller k-acceptor $\mathcal{A}=(\mathcal{A},c)$ the structure $(C_{\mathcal{A}};\leq_0,\leq_1,c)$ where $C_{\mathcal{A}}$ is the set of cycles of $\mathcal{A}$, $D\leq_0 E$ iff some state in D is reachable in the graph of the automaton $\mathcal{A}$ from some state in E, and $D\leq_1 E$ iff $D\subseteq E$.

4) The structure $(C_{\mathcal{A}}; \leq_0, \leq_1, c)$ may be identified with some $P_{\mathcal{A}} \in \mathcal{P}_k(2)$.
5) Using the known facts [S98] that $(\mathbf{\Sigma}^0_1 \cap \mathcal{R}, \mathbf{\Sigma}^0_2 \cap \mathcal{R})$ have the reduction property conclude that $\mathbf{\Sigma}\mathcal{R}(P_{\mathcal{A}}) = \mathbf{\Sigma}\mathcal{R}_{red}(F_{\mathcal{A}})$ where $F_{\mathcal{A}} \in \mathcal{T}_k^{\sqcup}(2)$ is the natural unfolding of $P_{\mathcal{A}}$.
6) Check that $L(\mathcal{A}, c)$ is CA-complete in $\mathbf{\Sigma}(F_{\mathcal{A}})$ and DA-complete in $\mathbf{\Sigma}\mathcal{R}(F_{\mathcal{A}})$ and conclude that $L(\mathcal{A}, c) \equiv_{DA} r(F_{\mathcal{A}})$.

So far, our results for ω-regular partitions generalized the corresponding results for ω-regular languages. Now we show that the structure of ω-regular languages is indeed much simpler than that of ω-regular k-partitions for $k > 2$. Recall that *first-order theory* $FO(A)$ of a structure A of signature σ is the set of first-order σ-sentences of signature which are true in A. Using the main result above, we show in [KS07]:

Theorem

For any $k \geq 3$, $FO(\mathcal{T}_k^{\sqcup}(2); \leq_h)$ is undecidable and, moreover, is computably isomorphic to the first-order arithmetic $FO(\omega; +, \cdot)$. In contrast, $FO(\mathcal{T}_2^{\sqcup}(2); \leq_h)$ is decidable.

Also, for $k \geq 3$ the automorphism group of $(\mathcal{T}_k^{\sqcup}(2); \leq_h)$ is isomorphic to the symmetric group on k elements.

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