Logic, Complexity, and Infinite Computations

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Complexity of finite computations

Complexity of finite computations is often measured by the amount of time or space needed to accept a word of length n.

P = DTIME(Pol)

NP = NTIME(Pol)

P = NP ?

Languages of finite words accepted by different finite machines

- A regular language (accepted by a finite automaton) is in the class **DTIME(n)**.
- A 1-counter language or a context-free language is in the class **DTIME (n^3)**.
- There are recursive languages, accepted by Turing machines, in the class **DTIME(2^n)$\setminus$ P**.
- There are recursive languages, accepted by Turing machines, which are non elementary. For instance Büchi's procedure (1962) to decide whether a monadic second order formula of size n of $S1S$ is true in the structure $(\omega, <)$ might run in time $\underbrace{2^{2^{\cdot^{\cdot^{2^n}}}}}_{\mathbf{O(n)}}$, Moreover Meyer (1975) proved that one cannot essentially improve this result: the monadic second order theory of $(\omega, <)$ is not elementary recursive.

Acceptance of infinite words

- **In the sixties**,
  Acceptance of infinite words by finite automata was firstly considered by **Büchi** in order to study the decidability of the monadic second order theory S1S of one successor over the integers.
- Since then ω-regular languages accepted by Büchi automata and their extensions have been much studied and used for **specification and verification of non terminating systems**.

Büchi acceptance condition

An automaton $\mathcal{A}$ reading infinite words over the alphabet Σ is equipped with a finite set of states K and a set of final states $F \subseteq K$.

A run of $\mathcal{A}$ reading an infinite word $\sigma \in \Sigma^\omega$ is said to be accepting iff there is some state $q_f \in F$ appearing infinitely often during the reading of σ.

An infinite word $\sigma \in \Sigma^\omega$ is accepted by $\mathcal{A}$ if there is (at least) one accepting run of $\mathcal{A}$ on σ.

An ω-language $L \subseteq \Sigma^\omega$ is accepted by $\mathcal{A}$ if it is the set of infinite words $\sigma \in \Sigma^\omega$ accepted by $\mathcal{A}$.

Context free or regular ω-languages

(Cohen and Gold 1977; Linna 1976)

Let $L \subseteq \Sigma^\omega$. Then the following propositions are equivalent :

- L is accepted by a Büchi pushdown automaton.
- L is accepted by a Muller pushdown automaton.
- $L = \bigcup_{1 \leq i \leq n} U_i . V_i^\omega$,
  for some context free finitary languages U_i and V_i.
- L is a context free ω-language.

A similar theorem holds if we:

- omit the pushdown stack and replace context free by regular,
- or replace pushdown and context-free by 1-counter.

Complexity of ω-languages

The question naturally arises of the complexity of ω-languages accepted by various kinds of automata.

A way to study the complexity of ω-languages is to consider their topological complexity.

Topology on Σ^ω

The natural prefix metric on the set Σ^ω of ω-words over Σ is defined as follows:

For $u, v \in \Sigma^\omega$ and $u \neq v$ let

$$\delta(u, v) = 2^{-n}$$

where n is the least integer such that:

the $(n+1)^{st}$ letter of u is different from the $(n+1)^{st}$ letter of v.

This metric induces on Σ^ω the usual Cantor topology for which :

- *open subsets* of Σ^ω are in the form $W.\Sigma^\omega$, where $W \subseteq \Sigma^\star$.
- *closed subsets* of Σ^ω are complements of *open subsets* of Σ^ω.

Borel Hierarchy

Σ_1^0 is the class of open subsets of Σ^ω,

Π_1^0 is the class of closed subsets of Σ^ω,

for any integer $n \geq 1$:

Σ_{n+1}^0 is the class of countable unions of Π_n^0-subsets of Σ^ω.

Π_{n+1}^0 is the class of countable intersections of Σ_n^0-subsets of Σ^ω.

Π_{n+1}^0 is also the class of complements of Σ_{n+1}^0-subsets of Σ^ω.

Borel Hierarchy

The Borel hierarchy is also defined for levels indexed by countable ordinals.

For any countable ordinal $\alpha \geq 2$:

Σ^0_α is the class of countable unions of subsets of Σ^ω in $\bigcup_{\gamma<\alpha} \Pi^0_\gamma$.

Π^0_α is the class of complements of Σ^0_α-sets

$\Delta^0_\alpha = \Pi^0_\alpha \cap \Sigma^0_\alpha$.

Borel Hierarchy

Below an arrow $\rightarrow$ represents a strict inclusion between Borel classes.

$$
\begin{array}{ccccccccccc}
 & & \Pi_1^0 & & & & & \Pi_\alpha^0 & & \Pi_{\alpha+1}^0 & \\
 & \nearrow & & \searrow \;\; \nearrow & & & \nearrow & & \searrow \;\; \nearrow & & \\
\Delta_1^0 & & & & \Delta_2^0 & \cdots\;\cdots & \Delta_\alpha^0 & & \Delta_{\alpha+1}^0 & & \cdots \\
 & \searrow & & \nearrow \;\; \searrow & & & \searrow & & \nearrow \;\; \searrow & & \\
 & & \Sigma_1^0 & & & & & \Sigma_\alpha^0 & & \Sigma_{\alpha+1}^0 &
\end{array}
$$

A set $X \subseteq \Sigma^\omega$ is a Borel set iff it is in $\bigcup_{\alpha<\omega_1} \Sigma_\alpha^0 = \bigcup_{\alpha<\omega_1} \Pi_\alpha^0$ where ω_1 is the first uncountable ordinal.

Beyond the Borel Hierarchy

There are some subsets of Σ^ω which are not Borel. Beyond the Borel hierarchy is the projective hierarchy.

The class of Borel subsets of Σ^ω is strictly included in the class Σ^1_1 of analytic sets which are obtained by projection of Borel sets.

A set $E \subseteq \Sigma^\omega$ is in the class Σ^1_1 iff :

$\exists F \subseteq (\Sigma \times \{0, 1\})^\omega$ such that F is Π^0_2 and

E is the projection of F onto Σ^ω

A set $E \subseteq \Sigma^\omega$ is in the class Π^1_1 iff $\Sigma^\omega - E$ is in Σ^1_1.

Suslin's Theorem states that : Borel sets = $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$

Complete Sets

A set $E \subseteq \Sigma^\omega$ **is $\mathcal{C}$-complete**, where $\mathcal{C}$ is a Borel class Σ^0_α or Π^0_α

or the class Σ^1_1, **for reduction by continuous functions** iff :

$\forall F \subseteq \Gamma^\omega \quad F \in \mathcal{C}$ iff :

$\exists f$ continuous, $f : \Gamma^\omega \to \Sigma^\omega$ such that $F = f^{-1}(E)$

$(x \in F \leftrightarrow f(x) \in E)$.

Example : $\{\sigma \in \{0,1\}^\omega \mid \exists^\infty i \;\; \sigma(i) = 1\}$ is a Π^0_2-complete-set and it is accepted by a deterministic Büchi automaton.

More Examples of Complete Sets

Examples :

$\{\sigma \in \{0,1\}^\omega \mid \exists i \;\; \sigma(i) = 1\}$ is a **Σ^0_1-complete-set**.

$\{\sigma \in \{0,1\}^\omega \mid \forall i \;\; \sigma(i) = 1\} = \{1^\omega\}$ is a **Π^0_1-complete-set**.

$\{\sigma \in \{0,1\}^\omega \mid \exists^{<\infty} i \;\; \sigma(i) = 1\}$ is a **Σ^0_2-complete-set**.

All these ω-languages are ω-regular.

Complexity of ω-languages of deterministic machines

deterministic finite automata (Landweber 1969)

- ω-**regular languages** accepted by **deterministic Büchi automata** are Π^0_2-sets.
- ω-**regular languages** are boolean combinations of Π^0_2-sets hence Δ^0_3-sets.

deterministic Turing machines

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Complexity of ω-Languages of Non Deterministic Turing Machines

Non deterministic Büchi or Muller Turing machines accept **effective analytic sets** (Staiger). The class **Effective-Σ^1_1** of **effective analytic sets** is obtained as the class of **projections of arithmetical sets** and **Effective-Σ^1_1** $\subsetneq \Sigma^1_1$.

Let ω_1^{CK} be the first non recursive ordinal.

Topological Complexity of Effective Analytic Sets

- There are some Σ^1_1-complete sets in **Effective-Σ^1_1**.
- For every non null ordinal $\alpha < \omega_1^{\mathrm{CK}}$, there exists some Σ^0_α-complete and some Π^0_α-complete ω-languages in the class **Effective-Σ^1_1**.
- **(Kechris, Marker and Sami 1989)**
  The supremum of the set of Borel ranks of **Effective-Σ^1_1-sets** is a countable ordinal $\gamma^1_2 > \omega_1^{\mathrm{CK}}$.

Topological complexity of 1-counter or context free ω-languages

Let $1 - CL_\omega$ be the class of real-time 1-counter ω-languages.

Let $\mathcal{C}$ be a class of ω-languages such that:

$$1 - CL_\omega \subseteq \mathcal{C} \subseteq \textbf{Effective-}\Sigma^1_1.$$

(a) **(F. and Ressayre 2003)** There are some Σ^1_1-complete sets in **the class** $\mathcal{C}$.

(b) **(F. 2005)** The Borel hierarchy of **the class** $\mathcal{C}$ is equal to the Borel hierarchy of **the class Effective-Σ^1_1.**

(c) γ^1_2 is the supremum of the set of Borel ranks of ω-languages in **the class** $\mathcal{C}$.

(d) For every non null ordinal $\alpha < \omega_1^{\mathrm{CK}}$, there exists some Σ^0_α-complete and some Π^0_α-complete ω-languages in **the class** $\mathcal{C}$.

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Topological complexity of 1-counter or context free ω-languages

Theorem (F. 2005)

The Wadge hierarchy of the class of ω-languages accepted by real-time 1-counter Büchi automata is equal to the Wadge hierarchy of the class of ω-languages of Büchi Turing machines.

Sketch of the proof

It is well known that every Turing machine can be simulated by a (non real time) 2-counter automaton.

We denote **BCL**$(2)_\omega$ the class of ω-languages accepted by Büchi 2-counter automata.

Thus the topological complexity of ω-languages in the class **BCL**$(2)_\omega$ is equal to the topological complexity of ω-languages accepted by Büchi Turing machines.

Sketch of the proof

First, from a 2-counter automaton A accepting an ω-language $L \subseteq X^\omega$, we construct a real-time 8-counter Büchi automaton B accepting an ω-language of the same topological complexity.

First, we add a storage type called a queue to a 2-counter Büchi automaton in order to read ω-words in real-time.

Then the queue can be simulated by

- two pushdown stacks or
- four counters,
  because each pushdown stack may be simulated by two counters.

Sketch of the proof

This simulation is not done in real-time but one can bound the number of transitions needed to simulate the queue. This allows to pad the strings in L with enough extra letters so that the new language $\theta_S(L)$ will be read in real-time by a 8-counter Büchi automaton.

The padding is obtained via the function $\theta_S : X^\omega \to (X \cup \{E\})^\omega$, where $S = (3k)^3$, with $k = card(X) + 2$, and for all $x \in X^\omega$:

$$\theta_S(x) = x(1).E^S.x(2).E^{S^2}.x(3).E^{S^3}.x(4) \ldots x(n).E^{S^n}.x(n+1).E^{S^{n+1}} \ldots$$

The ω-language $\theta_S(L)$ is accepted in real time by a Büchi automaton with $2 + 4 + 2 = 8$ counters.

Sketch of the proof

The next step is to simulate a *real-time* 8-counter Büchi automaton $\mathcal{A}$, by a *real-time* 1-counter Büchi automaton $\mathcal{B}$.

The eight first prime numbers are $2; 3; 5; 7; 11; 13; 17; 19$.

We code the content $(c_1, c_2, \ldots, c_8)$ of eight counters by the product $2^{c_1} \times 3^{c_2} \times \ldots \times (17)^{c_7} \times (19)^{c_8}$.

Then we code ω-words in $Y = X \cup \{E\}$ by ω-words in $Z = Y \cup \{A, B, 0\}$.

The new ω-words will have a **special shape** which will allow the propagation of the values of the counters of $\mathcal{A}$.

Sketch of the proof

The product of the eight first prime numbers is:

$$K = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 = 9699690$$

An ω-word $x \in Y^\omega$ is coded by the ω-word

$$h(x) = A.0^K.x(1).B.0^{K^2}.A.0^{K^2}.x(2).B.\ldots B.0^{K^n}.A.0^{K^n}.x(n).B\ldots$$

If $L(\mathcal{A}) \subseteq Y^\omega$ is accepted by a real time 8-counter Büchi automaton $\mathcal{A}$, then one can construct from $\mathcal{A}$ a 1-counter Büchi automaton $\mathcal{B}$, reading words over $Y \cup \{A, B, 0\}$, such that:

$$\forall x \in Y^\omega \quad h(x) \in L(\mathcal{B}) \longleftrightarrow x \in L(\mathcal{A})$$

Sketch of the proof

The mapping $h : Y^\omega \rightarrow (Y \cup \{A, B, 0\})^\omega$ is continuous.

The complement $h(Y^\omega)^-$ of the ω-language $h(Y^\omega)$ is an open subset of $(Y \cup \{A, B, 0\})^\omega$ and is accepted by a real time 1-counter automaton.

Thus the ω-language

$$h(L(\mathcal{A})) \cup h(Y^\omega)^- = L(\mathcal{B}) \cup h(Y^\omega)^-$$

is in the class $\mathbf{BCL}(1)_\omega$ and it has the same topological complexity as the ω-language $L(\mathcal{A})$.

Decision Problems

Castro and Cucker proved (1989) that many decision problems about ω-languages of Turing machines are highly undecidable, i.e.

located beyond the arithmetical hierarchy.

From their results and from the previous constructions, we can show that some decision problems about ω-languages of 1-counter automata are also highly undecidable.

Some Decision Problems

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two 1-counter automata over the alphabet Σ. Can we decide whether

- $L(\mathcal{C}_1)$ is empty ?
- $L(\mathcal{C}_1)$ is infinite ?
- $L(\mathcal{C}_1) = \Sigma^\omega$?
- $L(\mathcal{C}_1) = L(\mathcal{C}_2)$?
- $L(\mathcal{C}_1) \subseteq L(\mathcal{C}_2)$?
- $L(\mathcal{C}_1)$ is unambiguous ?
- $L(\mathcal{C}_1)$ is Borel ?
- ...

Some differences between Turing machines and 1-counter automata

Theorem (Castro and Cucker 1989)

The non-emptiness problem and the infiniteness problem for ω-languages of Turing machines are Σ_1^1-complete.

Theorem (Cohen and Gold 1977)

The non-emptiness problem and the infiniteness problem for ω-languages of 1-counter Büchi automata are decidable.

Proof. An ω-language L is accepted by a 1-counter Büchi automaton iff it is of the form $L = \bigcup_{1 \leq i \leq n} U_i.V_i^\omega$, for some 1-counter finitary languages U_i and V_i. The emptiness problem for 1-counter (and even context-free) finitary languages is decidable.

Some similarities

Theorem (Castro and Cucker 1989; F. 2009)

The following problems are Π^1_2-complete for ω-languages of Turing machines and for ω-languages of 1-counter Büchi automata:

1. *The universality problem.*
2. *The inclusion problem.*
3. *The equivalence problem.*
4. *The cofiniteness problem.*

Some undecidable problems higher in the analytical hierarchy

Some decisions problems for ω-languages of Turing machines and for ω-languages of 1-counter Büchi automata are located

above the two first levels of the analytical hierarchy.

We can use Set Theory to obtain such lower bounds of decision problems.

Perfect Sets, Thin Sets

Definition

Let $P \subseteq \Sigma^\omega$, where Σ is a finite alphabet having at least two letters. The set P is a perfect subset of Σ^ω iff it is a non-empty closed set which has no isolated points.

A perfect subset of Σ^ω has cardinality $2^{\aleph_0}$.

Definition

A set $X \subseteq \Sigma^\omega$ is said to be thin iff it contains no perfect subset.

Theorem (Souslin)

*(**ZFC**) An analytic set $X \subseteq \Sigma^\omega$ is either countable or contains a perfect subset. Thus every thin analytic set is countable.*

This result is not true for co-analytic sets in **ZFC**. We need additional axioms like analytic determinacy.

The constructible sets

The class **L** of *constructible sets* in a model **V** of **ZF** is defined by

$$\mathbf{L} = \bigcup_{\alpha \in \mathbf{ON}} \mathbf{L}(\alpha)$$

where the sets $\mathbf{L}(\alpha)$ are constructed by induction as follows:

1. $\mathbf{L}(0) = \emptyset$
2. $\mathbf{L}(\alpha) = \bigcup_{\beta<\alpha} \mathbf{L}(\beta)$, for α a limit ordinal, and
3. $\mathbf{L}(\alpha+1)$ is the set of subsets of $\mathbf{L}(\alpha)$ which are definable from a finite number of elements of $\mathbf{L}(\alpha)$ by a first-order formula relativized to $\mathbf{L}(\alpha)$.

If **V** is a model of **ZF** and **L** is the class of *constructible sets* of **V**, then the class **L** forms a model of **ZFC + CH.** Notice that the axiom (**V=L**) means "every set is constructible" and that it is consistent with **ZFC.**

The Largest Thin Effective Coanalytic Set

Theorem (Kechris 1975; Guaspari, Sacks)

*(**ZFC**) Let Σ be a finite alphabet having at least two letters. There exists a thin Π_1^1-set $\mathcal{C}_1(\Sigma^\omega) \subseteq \Sigma^\omega$ which contains every thin, Π_1^1-subset of Σ^ω. It is called the largest thin Π_1^1-set in Σ^ω.*

Theorem (Kechris 1975; Guaspari, Sacks)

*(**ZFC**) The cardinal of the largest thin Π_1^1-set in Σ^ω is equal to the cardinal of $\omega_1^{\mathbf{L}}$.*

This means that in a given model **V** of **ZFC** the cardinal of the largest thin Π_1^1-set in Σ^ω is equal to the cardinal *in* **V** of the ordinal $\omega_1^{\mathbf{L}}$ which plays the role of the cardinal $\aleph_1$ in the inner model **L** of constructible sets of **V**.

$$\omega_1^{\mathbf{L}} \leq \omega_1$$

Theorem

1. *(**ZFC + V=L**) The largest thin Π^1_1-set in Σ^ω is not a Borel set.*
2. *(**ZFC** + $\omega_1^{\mathbf{L}} < \omega_1$) The largest thin Π^1_1-set in Σ^ω is countable, hence a Σ^0_2-set.*

Proof. In (**ZFC + V=L**) it holds that $\omega_1 = \omega_1^{\mathbf{L}}$. Thus the set $\mathcal{C}_1(\Sigma^\omega)$ has cardinal ω_1 and it is not countable. But it is thin, hence has no perfect subset. Thus it cannot be a Borel set because Borel sets have the perfect set property.

(**ZFC** + $\omega_1^{\mathbf{L}} < \omega_1$) the ordinal $\omega_1^{\mathbf{L}}$ is countable so the set $\mathcal{C}_1(\Sigma^\omega)$ is countable. It is a countable union of singletons, and each singleton is a closed set. Thus $\mathcal{C}_1(\Sigma^\omega)$ is a countable union of closed sets, i.e. a Σ^0_2-set.

From effective coanalytic sets to 1-counter automata

The complement of $\mathcal{C}_1(\Sigma^\omega) \subseteq \Sigma^\omega$ is an effective analytic set accepted by a Büchi Turing machine $\mathcal{T}$.

We can now use previous constructions to obtain:

- A 2-counter Büchi automaton $\mathcal{A}_1$,
- A real time 8-counter Büchi automaton $\mathcal{A}_2$,
- A real time 1-counter Büchi automaton $\mathcal{A}_3$,

such that $L(\mathcal{T})$, $L(\mathcal{A}_1)$, $L(\mathcal{A}_2)$, and $L(\mathcal{A}_3)$, all have the same toplogical complexity.

The Topological complexity of a 1-counter ω-language depends on the models of ZFC

Theorem (F. 2009)

There exists a 1-counter Büchi automaton $\mathcal{A}$ such that the topological complexity of the ω-language $L(\mathcal{A})$ is not determined by the axiomatic system **ZFC**.

1. (**ZFC + V=L**). *The ω-language $L(\mathcal{A})$ is a true analytic set.*
2. (**ZFC** $+ \omega_1^{\mathbf{L}} < \omega_1$). *The ω-language $L(\mathcal{A})$ is a Π_2^0-set.*

Infinitary rational relations

(Gire 1981, Gire and Nivat 1984)
A set $R \subseteq \Sigma^\omega \times \Gamma^\omega$ is an infinitary rational relation iff one the two following equivalent conditions holds :

- R is recognized by a Büchi transducer $\mathcal{T}$:

  R is the set of pairs $(u, v) \in \Sigma^\omega \times \Gamma^\omega$ such that u is the input word and v is the output word of a successful computation of $\mathcal{T}$.
- R is accepted by a 2-tape Büchi automaton $\mathcal{A}$ with two asynchronous reading heads.

Similar results for 2-tape Büchi automata

Infinitary rational relations have same topological complexity as ω-languages accepted by real-time 1-counter Büchi automata or by Büchi Turing machines (i.e. effective analytic sets). And:

Theorem (F. 2009)

The topological complexity of an ω-language accepted by a 2-tape Büchi automaton is not determined by the axiomatic system **ZFC**. *Indeed there is a 2-tape Büchi automaton $\mathcal{B}$ such that:*

1. *There is a model V_1 of* **ZFC** *in which the ω-language $L(\mathcal{B})$ is an analytic but non Borel set.*
2. *There is a model V_2 of* **ZFC** *in which the ω-language $L(\mathcal{B})$ is a Π^0_2-set.*

PROGRAMS IN COMPUTER SCIENCE

Some programs make a computation, get a result, and then stop. Other ones have to maintain the good behaviour of a system:

- Operating systems (Internet)
- safety systems (power plant, . . .)
- aircraft autopilot

In particular, these systems are in relation with an environment, and must have the "good" response to any changes of the environment.

INFINITE GAMES

A system in relation with an environment may be specified by an infinite game between two players.

Two players:

- Player 1 : the computer program
- Player 2 : the environment

The possible actions of the players are represented by letters of a finite alphabet A.

INFINITE PLAY

The two players compose an infinite word over the alphabet A:

Player 1 : a_1 $\searrow$ a_3 $\searrow$ a_5 $\searrow$ $\ldots$

Player 2 : $\nearrow$ a_2 $\nearrow$ a_4 a_6

The infinite word $a_1.a_2.a_3\ldots$ **represents the infinite behaviour of the system.**

A good behaviour **is represented by a set of infinite words** $L \subseteq A^\omega$ **called the winning set for Player 1.**

The above game, with perfect information, is a Gale-Stewart game $G(L)$.

WINNING STRATEGIES

A strategy for Player 1 is a mapping $f : (A^2)^\star \longrightarrow A$. Player 1 follows the strategy f iff $\forall n \geq 1$: $a_{2n+1} = f(a_1 a_2 \ldots a_{2n})$.

The strategy f is winning for Player 1 if it ensures a good behaviour of the system, **i.e. such that :** the infinite word written by the two players belongs to the winning set L:

$$a_1.a_2.a_3 \ldots \in L$$

A winning strategy for Player 2 **is a strategy for Player 2 which ensures** that $a_1.a_2.a_3 \ldots \notin L$.

A Gale-Stewart game $G(L)$ is determined iff one of the two players has a winning strategy.

The important problems to solve in practice are:

(1) Is the game G(L) determined ?

(2) Which player has a winning strategy ?

(3) If Player 1 has a winning strategy, can we effectively construct this winning strategy ? Is it computable ?

(4) What is the complexity of this construction ? What are the necessary amounts of time and space ?

The winning set for Player 1 is often given as the set of infinite behaviours which satisfy a logical formula.

It is also often given as the set of infinite words accepted by a finite automaton, a one-counter automaton, a pushdown automaton, ... with a Büchi acceptance condition ...

Regular winning sets

Büchi and Landweber solved the famous Church's Problem posed in 1957, Rabin gave an alternative solution:

Theorem (Büchi-Landweber 1969; Rabin 1972)

If $L \subseteq \Sigma^\omega$ is a regular ω-language then:

- *The game $G(L)$ is determined.*
- *One can decide which Player has a winning strategy.*
- *On can construct effectively a winning strategy given by a finite state transducer.*

Deterministic context free winning sets

Walukiewicz extended this to the case of deterministic context free winning sets:

Theorem (Walukiewicz 1996)

If $L \subseteq \Sigma^\omega$ is a deterministic context free ω-language then:

- *The game $G(L)$ is determined.*
- *One can decide which Player has a winning strategy.*
- *On can construct effectively a winning strategy given by a pushdown transducer.*

Further extension to deterministic higher-order pushdown automata ([Cachat 2003], [Carayol, Hagues, Meyer, Ong, Serre 2008])

The question of the determinacy

The determinacy of regular or deterministic context-free games follows from the determinacy of Borel games. (Martin 1975).

The question remained open for non-deterministic pushdown automata, one-counter automata, 2-tape automata: these automata accept non-Borel sets.

Theorem (Martin 1970 and Harrington 1978)

The effective analytic determinacy is equivalent to the existence of a particular real called $0^\sharp$.
The existence of the real $0^\sharp$ is known in set theory to be a large cardinal assumption, and is not provable in **ZFC***.*

A set of ordinals C is a set of indiscernibles in the constructible universe $\mathbf{L}$ iff:

• For each first-order formula $\varphi(x_1, \ldots, x_n)$ in the language of set theory,

• For all finite sequences $\alpha_{i_1} < \alpha_{i_2} < \ldots < \alpha_{i_n}$ and $\beta_{i_1} < \beta_{i_2} < \ldots < \beta_{i_n}$ of ordinals in C, it holds that:

$$\mathbf{L} \models \varphi(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n}) \Longleftrightarrow \mathbf{L} \models \varphi(\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_n})$$

The real $0^\sharp$

The existence of the real $0^\sharp$ in a model **V** of **ZFC** is equivalent to the existence of an uncountable set of indiscernible ordinals in the constructible universe **L.**
(The existence of such a set was proven firstly by Silver from the existence of a Ramsey cardinal in 1966)

• The real $0^\sharp$ is the code in 2^ω of a set of integers, the set of Gödel numbers of formulas which are satisfied by an uncountable set of indiscernibles ordinals in **L.**

• The existence of the real $0^\sharp$ is equivalent to the existence of a non-trivial elementary embedding $j : \mathbf{L} \to \mathbf{L}$.

The context-free determinacy

Theorem (F. 2011)

The determinacy of games $G(L)$, where L is accepted by a real-time 1-counter Büchi automaton, is equivalent to the effective analytic determinacy, and thus it is not provable in ZFC.

Sketch of the proof

We start from an effective analytic set $L(\mathcal{T})$ accepted by a Büchi Turing machine $\mathcal{T}$.

Using some modifications of the previous constructions, we construct a real time 1-counter Büchi automaton $\mathcal{A}$ such that Player 1 (resp. Player 2) has a winning strategy in $G(L(\mathcal{T}))$ if and only if that Player 1 (resp. Player 2) has a winning strategy in the game $G(L(\mathcal{A}))$.

The game $G(L(\mathcal{T}))$ is determined iff the game $G(L(\mathcal{A}))$ is determined.

The context-free Wadge determinacy

Theorem (F. 2011)

The determinacy of Wadge games $W(L_1, L_2)$, where L_1 and L_2 are accepted by real-time 1-counter Büchi automata, is equivalent to the effective analytic determinacy, and thus it is not provable in ZFC.

Games with non-recursive strategies when they exist

Theorem (F. 2011)

There exists a 1-counter Büchi automaton $\mathcal{A}$ such that:

*(1) There is a model V_1 of **ZFC** in which Player 1 has a winning strategy σ in the game $G(L(\mathcal{A}))$. But σ cannot be recursive and not even hyperarithmetical.*

*(2) There is a model V_2 of **ZFC** in which the game $G(L(\mathcal{A}))$ is not determined.*

Moreover these are the only two possibilities: there are no models of **ZFC** in which Player 2 has a winning strategy.

Games with non-recursive strategies when they exist

Theorem (F. 2013)

There exists a real-time 1-counter Büchi automaton $\mathcal{A}$ such that the ω-language $L(\mathcal{A})$ is an arithmetical Δ_3^0-set and such that Player 2 has a winning strategy in the game $G(L(\mathcal{A}))$ but has no hyperarithmetical winning strategies in this game.

One cannot decide who wins a 1-counter game

Theorem (F. 2013)

There exists a recursive sequence of real time 1-counter Büchi automata $\mathcal{A}_n$, $n \geq 1$, such that all games $G(L(\mathcal{A}_n))$ are determined. But it is Π^1_2-complete (hence highly undecidable) to determine whether Player 1 has a winning strategy in the game $G(L(\mathcal{A}_n))$.

Games specified by 2-tape Büchi automata

The two players compose an infinite word over the alphabet $A \times B$:

Player 1 : (a_1, b_1) ↘ ↗ (a_3, b_3) ↘ ↗ (a_5, b_5) ↘ $\ldots$

Player 2 : (a_2, b_2) (a_4, b_4)

The infinite word $(a_1, b_1).(a_2, b_2).(a_3, b_3) \ldots \in (A \times B)^\omega$ **represents the infinite behaviour of the system.**

A good behaviour **is represented by a set of infinite words** $L(\mathcal{A}) \subseteq (A \times B)^\omega$ **accepted by a 2-tape Büchi automaton** $\mathcal{A}$**.**

The question of the determinacy

Theorem (F. 2012)

The determinacy of games $G(L)$, where L is accepted by a 2-tape (asynchronous) Büchi automaton, is equivalent to the effective analytic determinacy, and thus it is not provable in ZFC.

Sketch of the proof

We start from an ω-language accepted by a a real time 1-counter Büchi automaton $\mathcal{A}$.

We construct, from $\mathcal{A}$, a 2-tape Büchi automaton $\mathcal{B}$ such that Player 1 (resp. Player 2) has a winning strategy in $G(L(\mathcal{A}))$ if and only if Player 1 (resp. Player 2) has a winning strategy in the game $G(L(\mathcal{B}))$.

The game $G(L(\mathcal{A}))$ is determined iff the game $G(L(\mathcal{B}))$ is determined.

Games with non-recursive strategies

Theorem (F. 2012)

There exists a 2-tape Büchi automaton $\mathcal{A}$ such that:

(1) There is a model V_1 of **ZFC** *in which Player 1 has a winning strategy σ in the game $G(L(\mathcal{A}))$. But σ cannot be recursive and not even hyperarithmetical.*

(2) There is a model V_2 of **ZFC** *in which the game $G(L(\mathcal{A}))$ is not determined.*

The determinacy of Wadge games

Theorem (F. 2012)

The determinacy of Wadge games $W(L_1, L_2)$, where L_1, L_2 are accepted by 2-tape (asynchronous) Büchi automata, is equivalent to the effective analytic determinacy, and thus it is not provable in ZFC.

Games of maximum strength of determinacy

Theorem (F. 2012)

There exists a 1-counter Büchi automaton $\mathcal{A}_\sharp$ (resp., 2-tape Büchi automaton $\mathcal{B}_\sharp$) such that:
The game $G(L(\mathcal{A}_\sharp))$ (resp., $G(L(\mathcal{B}_\sharp))$) is determined if and only if the effective analytic determinacy holds.

A transfinite sequence of 2-tape Büchi automata

A transfinite sequence of games specified by 2-tape Büchi automata with increasing strength of determinacy.

Theorem (F. 2012)

There is a transfinite sequence of 2-tape Büchi automata $(\mathcal{A}_\alpha)_{\alpha<\omega_1^{\mathrm{CK}}}$, indexed by recursive ordinals, s.t.:

$$\forall \alpha < \beta < \omega_1^{\mathrm{CK}} \;\; [\; Det(G(L(\mathcal{A}_\beta))) \Longrightarrow Det(G(L(\mathcal{A}_\alpha))) \;]$$

but the converse is not true:

*For each recursive ordinal α there is a model $\mathbf{V}_\alpha$ of **ZFC** such that in this model the game $G(L(\mathcal{A}_\beta))$ is determined iff $\beta < \alpha$.*

Open Questions

Theorem [F. (2005)]
There are ω-languages accepted by Büchi 1-counter automata of every Borel rank of an effective analytic set.

Theorem [Kechris, Marker, and Sami (1989)]
The supremum of the set of Borel ranks of effective analytic sets is the ordinal $\gamma_2^1 > \omega_1^{\mathrm{CK}}$.

Every ω-language accepted by a Büchi 1-counter automaton can be written as a finite union $L = \bigcup_{1 \leq i \leq n} U_i.V_i^\omega$, **where for each integer i, U_i and V_i are 1-counter languages.**

Conjecture **From these results it seems plausible that there exist some ω-powers of languages accepted by 1-counter automata which have Borel ranks up to the ordinal γ_2^1, although these languages are located at the very low level in the complexity hierarchy of finitary languages.**

The ordinal γ_2^1 may depend on set theoretic axioms

The ordinal γ_2^1 is the least basis for subsets of ω_1 which are Π_2^1 in the codes.

It is the least ordinal such that whenever $X \subseteq \omega_1$, $X \neq \emptyset$, and $\hat{X} \subseteq WO$ is Π_2^1, there is $\beta \in X$ such that $\beta < \gamma_2^1$.

The least ordinal which is not a Δ_n^1-ordinal is denoted δ_n^1.

Theorem (Kechris, Marker and Sami 1989)

- (ZFC) $\delta_2^1 < \gamma_2^1$
- (V = L) $\gamma_2^1 = \delta_3^1$
- (Π_1^1-Determinacy) $\gamma_2^1 < \delta_3^1$

Are there effective analytic sets of every Borel rank $\alpha < \gamma_2^1$?

Open Questions

There is a 1-counter ω-language $L(\mathcal{A})$ which is Borel in some model of **ZFC** and non Borel in some other model of **ZFC**.

But

$$L(\mathcal{A}) = \bigcup_{1 \leq i \leq n} U_i.V_i^\omega$$

for some finitary 1-counter-languages U_i and V_i.

When $L(\mathcal{A})$ is non Borel then at least one ω-power language V_i^ω is non Borel.

Are all V_i^ω Borel in the other case ?

Does the topological complexity of the ω-power of a finitary 1-counter-language depend on the model of **ZFC**?

Open Questions

- Determine the Wadge hierarchy of **deterministic** infinitary rational relations.
- Determine the Wadge hierarchy of ω-languages accepted by non-deterministic 1-counter automata **without zero-test.**
- Study the **effectivity** of the Wadge hierarchy of deterministic context-free ω-languages, of some of its restrictions, of ω-languages of deterministic Petri nets.

THANK YOU !

The recursive ordinals form an initial segment of the countable ordinals.

The ordinals ω, ω^ω, ω^{ω^ω}, $\ldots$,

$$\varepsilon_0 = \lim_n \underbrace{\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}}_{n}$$

are recursive.

A set $R \subseteq \Sigma^\omega \times \Gamma^\omega$ is an infinitary rational relation iff it is generated from :

- the empty set $\emptyset$, and
- singletons $\{(a, \lambda)\}, \{(\lambda, b)\}$, $a \in \Sigma, b \in \Gamma$, where λ is the empty word.

by operations of

- finite union,
- concatenation product : $(u_1, v_1) \cdot (u_2, v_2) = (u_1 \cdot u_2, v_1 \cdot v_2)$
- star operation,
- operation $R \to R^\omega$ over finitary rational relations.

Notice that an infinitary rational relation $R \subseteq \Sigma^\omega \times \Gamma^\omega$ may be seen as an ω-language $R \subseteq (\Sigma \times \Gamma)^\omega$ over the alphabet $\Sigma \times \Gamma$.

The Analytical Hierarchy

Let $k, l > 0$ be some integers and $R \subseteq \mathcal{F}^k \times \mathbb{N}^l$, where $\mathcal{F}$ is the set of all mappings from $\mathbb{N}$ into $\mathbb{N}$.
The relation R is said to be recursive if its characteristic function is recursive.

A subset R of $\mathbb{N}^l$ is analytical if it is recursive or if there exists a recursive set $S \subseteq \mathcal{F}^m \times \mathbb{N}^n$, with $m \geq 0$ and $n \geq l$, such that $(x_1, \ldots, x_l)$ is in R iff

$$(Q_1 s_1)(Q_2 s_2) \ldots (Q_{m+n-l} s_{m+n-l}) S(f_1, \ldots, f_m, x_1, \ldots, x_n)$$

where Q_i is either $\forall$ or $\exists$ for $1 \leq i \leq m+n-l$, and where $s_1, \ldots, s_{m+n-l}$ are $f_1, \ldots, f_m, x_{l+1}, \ldots, x_n$ in some order.

$(Q_1 s_1)(Q_2 s_2) \ldots (Q_{m+n-l} s_{m+n-l}) S(f_1, \ldots, f_m, x_1, \ldots, x_n)$ is called a predicate form for R.
The reduced prefix is the sequence of quantifiers obtained by suppressing the quantifiers of type 0 from the prefix.

The Analytical Hierarchy

For $n > 0$, a Σ^1_n-prefix is one whose reduced prefix begins with $\exists^1$ and has $n-1$ alternations of quantifiers. For $n > 0$, a Π^1_n-prefix is one whose reduced prefix begins with $\forall^1$ and has $n-1$ alternations of quantifiers.

A Π^1_0-prefix or Σ^1_0-prefix is one whose reduced prefix is empty.

A predicate form is a Σ^1_n (Π^1_n)-form if it has a Σ^1_n (Π^1_n)-prefix.

The class of sets in $\mathbb{N}^l$ which can be expressed in Σ^1_n-form (respectively, Π^1_n-form) is denoted by Σ^1_n (respectively, Π^1_n).

The class $\Sigma^1_0 = \Pi^1_0$ is the class of arithmetical sets.